OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2016.03.29 **MAT462 Fonksiyonel Analiz II – Ara Sınavın Çözümleri** N. Course

Soru 1 (Strong and Weak Convergence of Operators). Let X and Z be Banach spaces. Let $A_n : X \to Z$ be a sequence of operators and let $A : X \to Z$ be an operator.

(a) [1p] Please write your student number at the top right of this page.

(b) [5p] Give the definition of *strong convergence* of A_n.

> We say that A_n converges strongly to A, and we write $\text{s-lim}_{n\to\infty} A_n = A$, if and only if
> $$A_n x \to Ax$$
> for all $x \in X$.

Suppose that

- $Y \subseteq X$;
- Y is dense in X;
- $A_n y \to Ay$ for all $y \in Y$;
- $\|A\| \leq C \in \mathbb{R}$; and
- $\|A_n\| \leq C$ for all $n \in \mathbb{N}$.

(c) [19p] Show that $\text{s-lim}_{n\to\infty} A_n = A$.

> Assume without loss of generality that $C > 0$.
> Let $\varepsilon > 0$. Let $x \in X$. Choose $y \in Y$ such that $\|x - y\| < \frac{\varepsilon}{3C}$. Since $A_n y \to Ay$, $\exists N \in \mathbb{N}$ such that
> $$n > N \implies \|A_n y - Ay\| < \frac{\varepsilon}{3}.$$
> But then
> $$\begin{aligned} n > N \implies \|A_n x - Ax\| &\leq \|A_n x - A_n y\| + \|A_n y - Ay\| + \|Ay - Ax\| \\ &\leq \|A_n\| \, \|x - y\| + \|A_n y - Ay\| + \|A\| \, \|y - x\| \\ &< C\frac{\varepsilon}{3C} + \frac{\varepsilon}{3} + C\frac{\varepsilon}{3C} = \varepsilon. \end{aligned}$$
> Therefore $A_n x \to Ax$ for all $x \in X$ and we are done.

(d) [5p] Give the definition of *weak convergence* of A_n.

> We say that A_n converges weakly to A, and we write $\text{w-lim}_{n\to\infty} A_n = A$, if and only if
> $$A_n x \rightharpoonup Ax$$
> for all $x \in X$.

Now suppose that

- $Y \subseteq X$;
- Y is dense in X;
- $A_n y \rightharpoonup Ay$ for all $y \in Y$;
- $\|A\| \leq C \in \mathbb{R}$; and
- $\|A_n\| \leq C$ for all $n \in \mathbb{N}$.

(e) [20p] Show that $\text{w-lim}_{n\to\infty} A_n = A$.

Let $\varepsilon > 0$. Let $x \in X$ and let $l \in Z^*$. Choose $y \in Y$ such that $\|x - y\| < \frac{\varepsilon}{3C\|l\|}$. Since $A_n y \rightharpoonup Ay$, $\exists N \in \mathbb{N}$ such that

$$n > N \implies |l(A_n y) - l(Ay)| < \frac{\varepsilon}{3}.$$

But then

$$\begin{aligned} n > N \implies |l(A_n x) - l(Ax)| &\leq |l(A_n x) - l(A_n y)| + |l(A_n y) - l(Ay)| + |l(Ay) - l(Ax)| \\ &\leq \|l\| \, \|A_n\| \, \|x - y\| + |l(A_n y) - l(Ay)| + \|l\| \, \|A\| \, \|y - x\| \\ &< \|l\| \, C \frac{\varepsilon}{3C\,\|l\|} + \frac{\varepsilon}{3} + \|l\| \, C \frac{\varepsilon}{3C\,\|l\|} = \varepsilon. \end{aligned}$$

Therefore $A_n x \rightharpoonup Ax$ for all $x \in X$ and we are done.

Soru 2 (Closed Operators).

(a) [1p] Please write your student number at the top right of this page.

(b) [5p] Give the definition of the *graph* of an operator.

Let $A : \mathfrak{D}(A) \subseteq X \to Y$ be an operator. The graph of A is

$$\Gamma(A) = \{(x, Ax) : x \in \mathfrak{D}(A)\}.$$

(c) [5p] Give the definition of a *closed* operator.

We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closed iff, its graph is a closed subset of $X \oplus Y$.

(d) [10p] Give an example of a closed operator. Prove that your operator is closed.

Many possible answers. E.g. identity operator on a Banach space.

(e) [10p] Give an example of an operator which is not closed. Prove that your operator is not closed.

Many possible answers. E.g. any unbounded operator defined on all of a Banach space.

Now let X and Y be normed spaces. Let $\mathfrak{D}(T) \subseteq X$ and let $T : \mathfrak{D}(T) \to Y$ be a bounded linear operator.

(f) [19p] Show that

$$\mathfrak{D}(T) \text{ is a closed subset of } X \implies T \text{ is a closed operator.}$$

Let $(x_n, Tx_n) \subseteq \Gamma(T)$ be a sequence such that $(x_n, Tx_n) \to (x, y) \in X \oplus Y$. Then $x_n \to x \in X$. Since $\mathfrak{D}(T)$ is a closed set, we know that $x \in \mathfrak{D}(T)$.

Moreover, since T is bounded (and therefore continuous), we know that $Tx_n \to Tx$. Therefore $\Gamma(T)$ is a closed set and hence T is a closed operator.

Soru 3 (Reflexive Spaces). Let X be a normed vector space. Define an operator J by

$$J(x)(l) = l(x)$$

for all $x \in X$ and $l \in X^*$.

(a) [1p] Please write your student number at the top right of this page.

(b) [10p] Fix $x_0 \in X$. Show that $J(x_0) \in X^{**}$.
[In other words: Show that $J(x_0) : X^* \to \mathbb{C}$ is bounded and linear]

Clearly

$$J(x_0)(\alpha l + \tilde{l}) = (\alpha l + \tilde{l})(x_0) = \alpha l(x_0) + \tilde{l}(x_0) = \alpha J(x_0)(l) + J(x_0)(\tilde{l})$$

for all $\alpha \in \mathbb{C}$ and for all $l, \tilde{l} \in X^*$. Thus $J(x_0)$ is linear.

Since

$$|J(x_0)(l)| = |l(x_0)| \leq \|l\| \, \|x_0\|$$

for all $l \in X^*$, we have that $\|J(x_0)\| \leq \|x_0\|$. Therefore $J(x_0)$ is bounded.

Hence $J(x_0) \in X^{**}$.

(c) [15p] Show that $J : X \to J(X)$ is an isomorphism.
[HINT: You must show that J is injective, that $J(\lambda x + y) = \lambda J(x) + J(y)$ $\forall \lambda, x, y$ and that $\|J(x)\| = \|x\|$ $\forall x$.]
[HINT: Use the Hahn-Banach Theorem or one of its corollaries for the final part of the previous hint.]

First, suppose that $J(x) = J(y)$. Then we have

$$l(x) = J(x)(l) = J(y)(l) = l(y)$$

for all $l \in X^*$, which implies that $x = y$. Hence J is an injection.

Clearly

$$J(\lambda x + y)(l) = l(\lambda x + y) = \lambda l(x) + l(y) = \lambda J(x)(l) + J(y)(l)$$

for all $l \in X^*$. So $J(\lambda x + y) = \lambda J(x) + J(y)$ for all $\lambda \in \mathbb{C}$ and for all $x, y \in X$.

Finally, it follows from the Hahn-Banach Theorem that for all $x \in X$, $\exists l_x \in X^*$ such that $\|l_x\| = 1$ and $l_x(x) = \|x\|$. So

$$|J(x)(l_x)| = |l_x(x)| = \|x\|$$

which, together with part (b) proves that $\|J(x)\| = \|x\|$ $\forall x$.

Therefore, J is an isomorphism.

(d) [9p] Give the definition of a reflexive space.

X is called reflexive if and only if $J(X) = X^{**}$

(e) [15p] Show that

$$X \text{ is reflexive } \implies X \text{ is complete.}$$

We proved in MAT461 Fonksiyonel Analiz I that if X is a normed space and Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space. Because $\mathbb{C}$ is complete, it follows that a dual space is always a Banach space.
Since X^{**} is the dual space of X^*, X^{**} is complete. But $J(X) = X^{**}$ because X is reflexive, so $J(X)$ is complete. Finally by part (d) we have that $X \cong J(X)$ is complete.