2017–05–23 **MAT462 Fonksiyonel Analiz II – Final Sınavın Çözümleri** N. Course

Soru 1 (Closed and Closable Operators).

(a) [5p] State the *Closed Graph Theorem*

Theorem (The Closed Graph Theorem). *Let $A : X \to Y$ be a linear map from a Banach space X to another Banach space Y. Then A is continuous if and only if its graph is closed.*

(b) [5p] Give the definition of a *closable operator*.

An operator A is called closable iff there exists an operator $\overline{A}$ such that $\overline{\Gamma(A)} = \Gamma(\overline{A})$. $\overline{A}$ is called the closure of A.

(c) [1p] Please write your student number at the top-right of this page.

Now suppose that

- A is a closable operator;
- $\overline{A}$ denotes the closure of A; and
- $\overline{A}$ is injective.

(d) [14p] Show that $\overline{A}^{-1} = \overline{A^{-1}}$.

We will define the notation

$$\Gamma^{-1} = \{(y, x) : (x, y) \in \Gamma\}.$$

It is then trivial to see that $\Gamma(A^{-1}) = \Gamma^{-1}(A)$. Then we have that

$$\overline{\Gamma(A^{-1})} = \overline{\Gamma^{-1}(A)} = \overline{\Gamma(A)}^{-1} = \Gamma^{-1}(\overline{A}) = \Gamma(\overline{A}^{-1}).$$

Soru 2 (The Hahn-Banach Theorem and Reflexivity). Let X be a Banach space. Consider the map $J : X \to X^{**}$ defined by $J(x)(l) = l(x)$.

Note that in the formula $J(x)(l) = l(x)$, we have $x \in X$.

(a) [2p] In the formula $J(x)(l) = l(x)$, what set is l in?

$l \in X^*$

(b) [2p] In the formula $J(x)(l) = l(x)$, what set is $J(x)$ in?

$J(x) \in X^{**}$

(c) [2p] In the formula $J(x)(l) = l(x)$, what set is $J(x)(l)$ in?

$J(x)(l) \in \mathbb{C}$

(d) [3p] Give the definition of a *reflexive space*.

X is called reflexive iff J is surjective.

(e) [5p] Show that J is injective.

Suppose that $x \neq y$. Clearly there exists $l \in X^*$ such that $l(x) \neq l(y)$. Then $J(x)(l) = l(x) \neq l(y) = J(y)(l)$ which implies that $J(x) \neq J(y)$. Therefore J is injective.

(f) [1p] Please write your student number at the top-right of this page.

(g) [5p] Show that $\|J(x)\|_{X^{**}} \leq \|x\|_X$ for all $x \in X$.

Since

$$|J(x)(l)| = |l(x)| \leq \|l\| \, \|x\|$$

we have that $\|J(x)\| \leq \|x\|$.

(h) [5p] Show that $\|J(x)\|_{X^{**}} \geq \|x\|_X$ for all $x \in X$.

Fix $x_0 \in X$. By the Hahn-Banach Theorem (there was a hint in the question name), there exists a linear functional l_0 which satisfies

- $\|l_0\| = 1$; and
- $l_0(x_0) = \|x_0\|$.

Hence $|J(x_0)(l_0)| = |l_0(x_0)| = \|x_0\|$ which implies that $\|J(x_0)\| \geq \|x_0\|$.

Soru 3 (Weak Convergence.). [25p] Please write two pages about *weak convergence*.

Soru 4 (The Baire Category Theorem and its Applications).

(a) [5p] Give the definition of a *nowhere dense* set.

A set is called nowhere dense iff its closure has empty interior.

(b) [7p] Give an example of a non-empty, nowhere dense set. You must prove that your set is nowhere dense.

An easy example would be a line in $\mathbb{R}^2$. Proof omitted.

(c) [5p] State the Open Mapping Theorem

Theorem (The Open Mapping Theorem). *Let $A \in \mathcal{B}(X,Y)$ be a bounded linear operator from one Banach space onto another. Then A is open (i.e. maps open sets to open sets).*

Theorem (The Inverse Mapping Theorem). *Let $A \in \mathcal{B}(X,Y)$ be a bounded linear bijection between Banach spaces. Then A^{-1} is continuous.*

(d) [8p] Prove the Inverse Mapping Theorem.
[HINT: Use the Open Mapping Theorem.]

Since A^{-1} is continuous $\iff$ A is open, the result follows by the Open Mapping Theorem.

Soru 5 (Hilbert-Schmidt Operators).

(a) [5p] Give the definition of the *Hilbert-Schmidt norm* and the definition of a *Hilbert-Schmidt operator*

The Hilbert-Schmidt norm of $K : X \to X$ is

$$\|K\|_2 = \left(\sum_j s_j(K)^2 \right)^{\frac{1}{2}}$$

where $s_j(K)$ are the singular values of K. A compact operator K is a Hilbert-Schmidt operator iff $\|K\|_2 < \infty$.

Now consider the operator $K : \ell^2(\mathbb{N}) \to \ell 2(\mathbb{N})$ defined by

$$(Kf)_n = \sum_{j=1}^{\infty} k_{n+j} f_j$$

where $k_j \in \mathbb{C}$ for all $j \in \mathbb{N}$. If this is unclear, I mean that

$$Kf = K(f_1, f_2, f_3, \ldots) = \left(\sum_{j=1}^{\infty} k_{1+j} f_j, \sum_{j=1}^{\infty} k_{2+j} f_j, \sum_{j=1}^{\infty} k_{3+j} f_j, \ldots \right).$$

Define a positive real number by

$$\lambda = \sum_{j=1}^{\infty} j \left| k_{j+1} \right|^2.$$

(b) [20p] Show that

$$K \text{ is a Hilbert-Schmidt operator} \qquad \iff \qquad \lambda < \infty$$

and show that $\|K\|_2 = \sqrt{\lambda}$ in this case.

By a Lemma from the course, a useful alternate formula for the Hilbert-Schmidt norm is

$$\|K\|_2 = \left(\sum_j \left\| K w^j \right\|^2 \right)^{\frac{1}{2}}$$

for any orthonormal basis $\{w^j\}$.I will use this formula with the orthonormal basis $\{\delta^j\}$. Since

$$(K\delta^p)_n = \sum_{j=1}^{\infty} k_{n+j}\delta_j^p = 0 + \ldots + 0 + k_{n+p} + 0 + \ldots = k_{n+p},$$

we have that

$$\|K\delta^p\|^2 = \sum_{n=1}^{\infty} \left|(K\delta^p)_n\right|^2 = \sum_{n=1}^{\infty} \left|k_{n+p}\right|^2$$

and that

$$\|K\|_2^2 = \sum_{p=1}^{\infty} \|K\delta^p\|^2 = \sum_{p=1}^{\infty}\sum_{n=1}^{\infty} \left|k_{n+p}\right|^2 = \left|k_2\right|^2 + 2\left|k_3\right|^2 + 3\left|k_4\right|^2 + \ldots = \lambda.$$

It is then trivial to see that K is Hilbert-Schmidt if and only if $\lambda < \infty$.