2017.03.28 **MAT462 Fonksiyonel Analiz II – Ara Sınavın Çözümleri** N. Course

Soru 1 (Closed Operators).

(a) [10p] Give the definition of a *closed* operator

> We say that an operator $A : \mathfrak{D}(A) \subseteq X \to Y$ is closed iff, its graph is a closed subset of $X \oplus Y$.

Now let $X = Y = \ell^2(\mathbb{N})$ with the norm $\|x\|_2 = \left(\sum_{j=1}^{\infty} |x_j|^2\right)^{\frac{1}{2}}$. Consider the operator $T : \mathcal{D}(T) \to \ell^2(\mathbb{R})$ where

$$\mathcal{D}(T) := \left\{a = (a_j)_{j=1}^{\infty} \in \ell^2(\mathbb{N}) : (ja_j)_{j=1}^{\infty} \in \ell^2(\mathbb{N})\right\} \subseteq \ell^1(\mathbb{N}) \subseteq \ell^2(\mathbb{N})$$

and

$$Tx := \|x\|_1 \delta^1 = \left(\sum_{j=1}^{\infty} |x_j|, 0, 0, 0, 0, 0, 0, \dots\right).$$

For example, if $x = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots, \frac{1}{j^2}, \dots\right)$ then $Tx = \left(\frac{\pi^2}{6}, 0, 0, 0, 0, 0, \dots\right)$ because $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$.

HINT 1: We proved in class that $\mathcal{D}(T)$ is a closed set – you may assume this.
HINT 2: You may also assume without proof that $\mathcal{D}(T)$ satisfies the following property:

$$(a^n)_{n=1}^{\infty} \subseteq \mathcal{D}(T) \text{ and } a^n \to 0 \implies \|a^n\|_1 \to 0.$$

(b) [10p] Use hint 2 to prove that if $(a^n) \subseteq \mathcal{D}(T)$, then

$$a^n \to a \implies \|a^n\|_1 \to \|a\|_1 .$$

> If $a^n \to a$, then $a^n - a \to 0$ which implies that $\|a^n - a\|_1 \to 0$. Therefore
>
> $$\|a^n\|_1 \leq \|a^n - a\|_1 + \|a\|_1 \to 0 + \|a\|_1$$
>
> by the triangle inequality.

(c) [1p] Please write your student number on this page.

(d) [7p] Show that if (a^n, Ta^n) is a Cauchy sequence in $\Gamma(T)$, then a^n is convergent in $\mathcal{D}(T)$.

> Clearly if (a^n, Ta^n) is Cauchy, then a^n is Cauchy in the Banach space $\ell^2(\mathbb{N})$. So $a^n \to a$. Since $\mathcal{D}(T)$ is closed (by the first hint), we must have that $a \in \mathcal{D}(T)$.

(e) [22p] Show that T is a closed operator.

We must show that $\Gamma(T)$ is a closed set.

Let (a^n, Ta^n) be a Cauchy sequence in $\Gamma(T)$. Then a^n and Ta^n are Cauchy sequences in the Banach space $\ell^2(\mathbb{N})$. By completeness, there exist $a, b \in \ell^2(\mathbb{N})$ such that $a^n \to a \in \mathcal{D}(T)$ and $Ta^n \to b$. It remains to prove that $Ta = b$.

By part (b) we have that

$$\|Ta^n - Ta\|_2^2 = \sum_{j=1}^{\infty} |(Ta^n)_j - (Ta)_j|^2 = |(Ta^n)_1 - (Ta)_1|^2 = |\|a^n\|_1 - \|a\|_1|^2 \to 0.$$

Therefore $Ta = b$.

Soru 2 (The Hahn-Banach Theorem).

(a) [10p] This course is called Functional Analysis/Fonksiyonel Analiz: What is a *functional*?

A functional is a (linear) function from a vector space to the field underlying the vector space.

Now let $X = \ell^\infty(\mathbb{N})$ with the norm $\|x\|_\infty = \sup_{j \in \mathbb{N}} |x_j|$. Consider the subspace

$$\mathfrak{c}(\mathbb{N}) := \left\{ a = (a_j)_{j=1}^\infty \in \ell^\infty(\mathbb{N}) : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} a_j \text{ exists} \right\} \subseteq \ell^\infty(\mathbb{N})$$

and the function $L : \mathfrak{c}(\mathbb{N}) \to \mathbb{C}$,

$$L(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k.$$

(b) [8p] Show that if $x = (x_j)_{j=1}^\infty$ is a convergent sequence in $\mathbb{C}$, then $x \in \mathfrak{c}(\mathbb{N})$ and $L(x) = \lim_{j\to\infty} x_j$.

Note that $L(\lambda) = \lambda$ for any constant sequence λ. Therefore, without loss of generality, we can suppose that $x_j \to 0$.

Let $\varepsilon > 0$. Then there exists N such that if $n > N$ then $|x_j| < \varepsilon$. Clearly $\frac{1}{n} \sum_{k=1}^{N} x_k \to 0$ as $n \to \infty$. Moreover

$$\left| \frac{1}{n} \sum_{k=N+1}^{n} x_k \right| \le \frac{1}{n} \sum_{k=N+1}^{n} |x_k| < \frac{1}{n} \sum_{k=N+1}^{n} \varepsilon = \frac{(n-N-1)\varepsilon}{n} < \varepsilon.$$

Therefore $\frac{1}{n} \sum_{k=1}^{n} x_k \to 0$ as $n \to \infty$. Hence $L(x) = 0 = \lim_{k\to\infty} x_k$ as required.

(c) [8p] Show that if $y \in \mathfrak{c}(\mathbb{N})$ then

$$L(Sy) = L(y)$$

where $(Sy)_n = y_{n+1}$ is the shift operator.

Clearly if $y \in \mathfrak{c}(\mathbb{N})$ then

$$L(Sy) - L(y) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} y_{k+1} - \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} y_k = \lim_{n\to\infty} \frac{y_{n+1} - y_1}{n} = 0.$$

(d) [8p] Show that the function $\varphi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$, $\varphi(x) = \|x\|_\infty$ is a convex function.

Clearly if $x, y \in \ell^\infty(\mathbb{N})$ then $\varphi(\lambda x + (1-\lambda)y) = \|\lambda x + (1-\lambda)y\| \leq \lambda \|x\|_\infty + (1-\lambda)\|y\|_\infty = \lambda\varphi(x) + (1-\lambda)\varphi(y)$ for all $\lambda \in (0,1)$ by the triangle inequality.

(e) [16p] Show that L can be extended to all of $\ell^\infty(\mathbb{N})$ such that

(a) L is linear; and

(b) $|L(x)| \leq \|x\|_\infty$

for all $x \in \ell^\infty(\mathbb{N})$.

And a very easy 10 points for you: Since L is a linear functional (you prove), defined on a subspace $Y = \mathfrak{c}(\mathbb{N})$, satisfying $|L(x)| \leq \varphi(x)$ for all $x \in \mathfrak{c}(\mathbb{N})$ (you prove), properties (i) and (ii) follow immediately from the Hahn-Banach Theorem.

Soru 3 (Strong Convergence of Operators). Let X and Y be Banach spaces. Let $A_n : X \to Y$ be a sequence of operators and let $A : X \to Y$ be an operator.

(a) [10p] Give the definition of *strong convergence* of A_n.

We say that A_n converges strongly to A, and we write $\operatorname{s-lim}_{n\to\infty} A_n = A$, if and only if

$$A_n x \to Ax$$

for all $x \in X$.

(b) [14p] Show that strong limits are unique. (Show that if $A = \operatorname*{s-lim}_{n\to\infty} A_n$ and $\tilde{A} = \operatorname*{s-lim}_{n\to\infty} A_n$ then $A = \tilde{A}$.)

This should be easy for you: Suppose that $A = \operatorname{s-lim}_{n\to\infty} A_n$ and $\tilde{A} = \operatorname{s-lim}_{n\to\infty} A_n$. Then $A_n x \to Ax$ and $A_n x \to \tilde{A}x$ for all $x \in X$. It follows that $(A - \tilde{A})x = \lim_{n\to\infty} Ax - \lim_{n\to\infty} Ax = 0$ for all $x \in X$. Therefore $A = \tilde{A}$.

Now suppose that

- $A_n, A \in \mathcal{B}(X, Y)$;
- $\operatorname*{s-lim}_{n\to\infty} A_n = A$;
- $x_n, x \in X$; and
- $\lim_{n\to\infty} x_n = x$;

(c) [13p] Show that if $\|A_n\| \leq C$ for all n, then $\lim_{n\to\infty} A_n x_n = Ax$.

We can calculate that

$$\begin{aligned} \|A_n x_n - Ax\| &\leq \|A_n x_n - A_n x\| + \|A_n x - Ax\| \leq \|A_n\| \|x_n - x\| + \|A_n x - Ax\| \\ &\leq C \|x_n - x\| + \|A_n x - Ax\| \to 0 \end{aligned}$$

since $x_n \to x$ and $A_n x \to Ax$. Therefore $\lim_{n\to\infty} A_n x_n = Ax$.

(d) [13p] Show that $\|A\| \leq \liminf_{n\to\infty} \|A_n\|$.

Choose $y \in X$ such that $\|y\| = 1$ and $Ay > \|A\| - \varepsilon$. Then

$$\|A\| - \varepsilon < Ay = \lim_{n\to\infty} A_n y = \liminf_{n\to\infty} A_n y \leq \liminf_{n\to\infty} \|A_n\|.$$

(This is almost exactly the same as the proof of Lemma 4.16(i).)