02.01.2013 **MAT 233 – Matematik III – Final Sınavın Çözümleri** N. Course

Question 1 (The Chain Rule and Directional Derivatives)**.**

(a) [14p] Suppose that $w = (x+y+z)^2$ where $x = r-s$, $y = \cos(r+s)$ and $z = \sin(r+s)$. **Use the Chain Rule** to calculate

$$\left.\frac{\partial w}{\partial r}\right|_{(r,s)=(1,-1)} \qquad \text{and} \qquad \left.\frac{\partial w}{\partial s}\right|_{(r,s)=(1,-1)}.$$

Notice first that $x|_{(r,s)=(1,-1)} = 1-(-1) = 2$, $y|_{(r,s)=(1,-1)} = \cos(1+(-1)) = \cos 0 = 1$ and $z|_{(r,s)=(1,-1)} = \sin(1+(-1)) = \sin 0 = 0$.
Since

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= 2(x+y+z)(1) - 2(x+y+z)\sin(r+s) + 2(x+y+z)\cos(r+s) \\ &= 2(x+y+z)(1-\sin(r+s)+\cos(r+s)),\end{aligned}$$

we can see that

$$\left.\frac{\partial w}{\partial r}\right|_{(r,s)=(1,-1)} = 2(2+1+0)(1-\sin 0+\cos 0) = 12.$$

Similarly

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= 2(x+y+z)(-1) - 2(x+y+z)\sin(r+s) + 2(x+y+z)\cos(r+s) \\ &= 2(x+y+z)(-1-\sin(r+s)+\cos(r+s)),\end{aligned}$$

so

$$\left.\frac{\partial w}{\partial s}\right|_{(r,s)=(1,-1)} = 2(2+1+0)(-1-\sin 0+\cos 0) = 0.$$

(b) [11p] Let $f(x,y) = 2xy - 3y^2$, $P_0 = (5,5)$ and $v = 4\mathbf{i} + 3\mathbf{j}$. Calculate the derivative of f at the point P_0 in the direction $\mathbf{v}$.
[HINT: $\mathbf{v}$ is not a unit vector.]

First note that $\|\mathbf{v}\| = \sqrt{4^2+3^2} = 5$. Define $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$. Then $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$.
We can calculate that

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = 2y\mathbf{i} + (2x-6y)\mathbf{j},$$

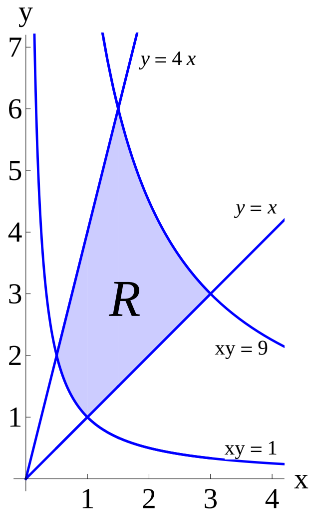
and

$$\nabla f|_{P_0} = 10\mathbf{i} - 20\mathbf{j}.$$

The derivative of f at the point P_0 in the direction $\mathbf{v}$ is

$$D_{\mathbf{u}}f|_{P_0} = \nabla f|_{P_0}\cdot\mathbf{u} = (10\mathbf{i} - 20\mathbf{j})\cdot\left(\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}\right) = 8 - 12 = -4$$

Question 2 (Substitutions in Multiple Integrals)**.** Let R be the region bounded by the curves $xy = 1$, $xy = 9$, $y = x$ and $y = 4x$.



[25p] Use the transformation $x = \frac{u}{v}$ and $y = uv$, to calculate

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy.$$

Since $x = \frac{u}{v}$ and $y = uv$, we have that $\frac{y}{x} = v^2$ and $xy = u^2$.
The Jacobian of this transformation is

$$J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = uv^{-1} + uv^{-1} = \frac{2u}{v}.$$

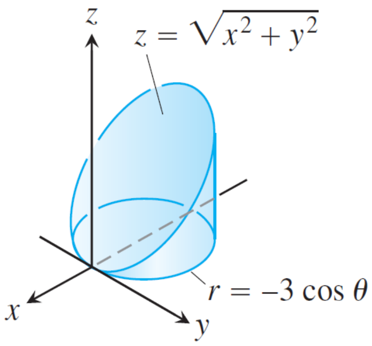
Next we need to look at the 4 edges of R:

$$\begin{aligned} y = x &\implies uv = \frac{u}{v} \implies v = 1 \\ y = 4x &\implies v = 2 \\ xy = 1 &\implies u = 1 \\ xy = 9 &\implies u = 3. \end{aligned}$$

Therefore

$$\begin{aligned} \iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy &= \int_{u=1}^{3} \int_{v=1}^{2} (v+u)\left(\frac{2u}{v}\right) dv\ du \\ &= \int_{u=1}^{3} \int_{v=1}^{2} 2u + \frac{2u^2}{v}\ dv\ du \\ &= \int_{u=1}^{3} \left[2uv + 2u^2 \log v\right]_{v=1}^{2}\ du \\ &= \int_{u=1}^{3} 2u + 2u^2 \log 2\ du \\ &= \left[u^2 + \frac{2}{3}u^2 \log 2\right]_1^3 \\ &= 8 + \frac{2}{3}(26)(\log 2) \\ &= 8 + \frac{52}{3}\log 2. \end{aligned}$$

Question 3 (Cylindrical Polar Coordinates)**.** Let $D \subseteq \mathbb{R}^3$ be the region shown below.



[25p] Calculate

$$\iiint_D z^2 \, dV.$$

[HINT: $\int_{\pi/2}^{\pi} \cos\theta d\theta = -1$, $\int_{\pi/2}^{\pi} \sin\theta d\theta = 1$, $\int_{\pi/2}^{\pi} \cos^2\theta d\theta = \frac{\pi}{4}$, $\int_{\pi/2}^{\pi} \sin^2\theta d\theta = \frac{\pi}{4}$, $\int_{\pi/2}^{\pi} \cos^3\theta d\theta = -\frac{2}{3}$, $\int_{\pi/2}^{\pi} \sin^3\theta d\theta = \frac{2}{3}$, $\int_{\pi/2}^{\pi} \cos^4\theta d\theta = \frac{3\pi}{16}$, $\int_{\pi/2}^{\pi} \sin^4\theta d\theta = \frac{3\pi}{16}$, $\int_{\pi/2}^{\pi} \cos^5\theta d\theta = -\frac{8}{15}$, $\int_{\pi/2}^{\pi} \sin^5\theta d\theta = \frac{8}{15}$, $\int_{\pi/2}^{\pi} \cos^6\theta d\theta = \frac{5\pi}{32}$. $\int_{\pi/2}^{\pi} \sin^6\theta d\theta = \frac{5\pi}{32}$.]

The region D is given by

$$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}, \qquad 0 \leq r \leq -3\cos\theta, \qquad 0 \leq z \leq \sqrt{x^2+y^2} = r.$$

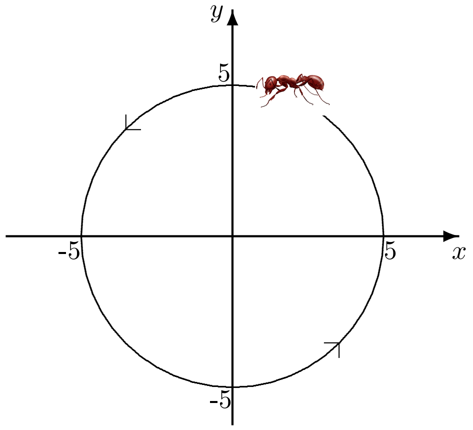
Therefore

$$\begin{aligned}
\iiint_D z^2 \, dV &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{-3\cos\theta} \int_0^r z^2 \, r dz dr d\theta \\
&= 2\int_{\frac{\pi}{2}}^{\pi} \int_0^{-3\cos\theta} \int_0^r z^2 \, r dz dr d\theta \\
&= 2\int_{\frac{\pi}{2}}^{\pi} \int_0^{-3\cos\theta} \left[\frac{1}{3} z^3 r\right]_{z=0}^{r} dr d\theta \\
&= 2\int_{\frac{\pi}{2}}^{\pi} \int_0^{-3\cos\theta} \frac{1}{3} r^4 \, dr d\theta \\
&= 2\int_{\frac{\pi}{2}}^{\pi} \left[\frac{1}{15} r^5\right]_0^{-3\cos\theta} d\theta \\
&= 2\int_{\frac{\pi}{2}}^{\pi} -\frac{243}{15} \cos^5\theta \, d\theta \\
&= -\frac{486}{15} \int_{\frac{\pi}{2}}^{\pi} \cos^5\theta \, d\theta \\
&= -\frac{486}{15}\left(-\frac{8}{15}\right) \qquad \text{(by the hint)} \\
&= \frac{2^4 \times 3^5}{3^2 \times 5^2} \\
&= \frac{2^4 \times 3^3}{5^2} \\
&= \frac{432}{25} \\
&= 17.28
\end{aligned}$$

Question 4 (Lagrange Multipliers)**.** The temperature (in °C) at a point (x, y) is given by

$$f(x, y) = 4x^2 - 4xy + y^2.$$

An ant walks around the circle of radius 5 centered at the origin.



[25p] What are the highest and lowest temperatures encountered by the ant?
[HINT: Use a Lagrange Multiplier.]

The gradient of f is $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = (8x - 4y)\mathbf{i} + (2y - 4x)\mathbf{j}$. Define a function $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(x, y) = x^2+y^2-25$. The circle is given by $g(x, y) = 0$. The gradient of g is $\nabla g = 2x\mathbf{i}+2y\mathbf{j}$.

We need to find x,y and λ such that $\nabla f = \lambda\nabla g$. We calculate that

$$(8x - 4y)\mathbf{i} + (2y - 4x)\mathbf{j} = \nabla f = \lambda\nabla g = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}.$$

So $8x - 4y = 2\lambda x$ and $2y - 4x = 2\lambda y$. Rearranging the latter equation gives

$$y = -\frac{2x}{\lambda - 1}$$

for $\lambda \neq 1$. Substituting into the former equation gives

$$8x + \frac{8x}{\lambda - 1} = 2\lambda x.$$

Therefore $x = 0$, $\lambda = 0$ or $\lambda = 5$. We look at these three cases separately.

CASE 1 ($x = 0$): If $x = 0$, then $y = 0$, but the point $(0, 0)$ is not on the circle $g(x, y) = 0$. Hence we know that $x \neq 0$.

CASE 2 ($\lambda = 0$): If $\lambda = 0$, then $y = 2x$. Then $0 = g(x, y) = g(x, 2x) = x^2 + 4x^2 - 25 \implies 5x^2 = 25 \implies x = \pm\sqrt{5}$ and $y = \pm 2\sqrt{5}$. We need to look at the two points: $(\sqrt{5}, 2\sqrt{5})$ and $(-\sqrt{5}, -2\sqrt{5})$.

CASE 3 ($\lambda = 5$): If $\lambda = 5$, then $y = -\frac{2x}{4} = -\frac{x}{2}$. Then $0 = g(x, y) = g(x, -\frac{x}{2}) = x^2 + \frac{x^2}{4} - 25 \implies \frac{5x^2}{4} = 25 \implies x = \pm 2\sqrt{5}$ and $y = \mp\sqrt{5}$. This gives us two more points to look at: $(2\sqrt{5}, -\sqrt{5})$ and $(-2\sqrt{5}, \sqrt{5})$.

Finally, we need to calculate f at these four points.

$$f(\sqrt{5}, 2\sqrt{5}) = 0 = f(-\sqrt{5}, -2\sqrt{5})$$
$$f(2\sqrt{5}, -\sqrt{5}) = 125 = f(-2\sqrt{5}, \sqrt{5}).$$

The highest temperature encountered by the ant is 125°C. The lowest temperature encountered by the ant is 0°C.

Question 5 (Extrema and Saddle Points).

(a) [10p] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 9x^3 + \frac{y^3}{3} - 4xy$. Find all the local maxima, local minima and saddle points of f.

Since f is defined, and is differentiable on all of $\mathbb{R}^2$, we only need to look at the points where $f_x = 0 = f_y$.

$$0 = \frac{\partial f}{\partial y} = y^2 - 4x \qquad \Longrightarrow \qquad x = \frac{y^2}{4}$$
$$0 = \frac{\partial f}{\partial x} = 27x^2 - 4y = \frac{27}{16}y^4 - 4y = y\left(\frac{27}{16}y^3 - 4\right) \qquad \Longrightarrow \qquad y = 0 \text{ or } y = \frac{4}{3}.$$

Since $y = 0 \implies x = 0$ and $y = \frac{4}{3} \implies x = \frac{y^2}{4} = \frac{4}{9}$, we must look at the points $(0, 0)$ and $(\frac{4}{3}, \frac{4}{9})$.

At $(0, 0)$, $f_{xx} = 54x = 0$, $f_{xy} = -4$ and $f_{yy} = 2y = 0$. So the Hessian is

$$H(f)(0, 0) = f_{xx}f_{yy} - f_{xy}^2\big|_{(0,0)} = -16 < 0.$$

Therefore f has a saddle point at $(0, 0)$.

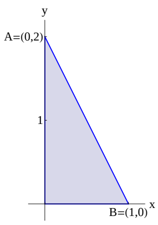
At $(\frac{4}{3}, \frac{4}{9})$, $f_{xx} = 54x = \frac{216}{3}$, $f_xy = -4$ and $f_{yy} = 2y = \frac{8}{9}$. The Hessian is

$$H(f)\left(\frac{4}{3}, \frac{4}{9}\right) = f_{xx}f_{yy} - f_{xy}^2\big|_{(\frac{4}{3},\frac{4}{9})} = \frac{216}{3}\frac{8}{9} - 16 = 64 > 0.$$

Since $f_{xx}|_{(\frac{4}{3},\frac{4}{9})} > 0$, the function f has a local minimum at $(\frac{4}{3}, \frac{4}{9})$.

Let R be the closed region bounded by the lines $x = 0$, $y = 0$ and $y + 2x = 2$ (see below). Let $g : R \to \mathbb{R}$ be defined by $g(x, y) = x^2 + y^2 - y$.

(b) [15p] Find the absolute maximum and absolute minimum of g on R.



First we look for critical points of g in the interior of R. But

$$0 = g_x = 2x \implies x = 0$$

and any point $(0, y)$ is not in the interior of R. This leaves the boundary of R to study.

On the corners: $g(0, 0) = 0$, $g(0, 2) = 0 + 4 - 2 = 2$ and $g(1, 0) = 1$.

On OA: Let $x = 0$. Define $h(y) = g(0, y) = y^2 - y$. Then $0 = h' = 2y - 1 \implies y = \frac{1}{2}$. We can calculate that $g(0, \frac{1}{2}) = h(\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} = -0.25$.

On OB: Let $y = 0$. Define $j(x) = g(x, 0) = x^2$. Then $0 = j' = 2x \implies x = 0$. We have already looked at $(0, 0)$.

On AB: Let $y = 2(1-x)$. Define $k(x) = g(x, 2(1-x)) = x^2+4(1-x)^2-2(1-x) = 5x^2-6x+2$. Then $0 = k' = 10x - 6 \implies x = 0.6 \implies y = 0.8$. We can calculate that $g(0.6, 0.8) = k(0.6) = 5\left(\frac{6}{10}\right)^2 - \frac{36}{10} + 2 = \frac{180-360+200}{100} = \frac{20}{100} = 0.2$

Therefore the absolute maximum of g on R is 2 and the absolute minimum is -0.25.