OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

15.11.2012 **MAT 233 – Matematik III – Ara Sınav** N. Course

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|---|---|
| ADI SOYADI | |
| ÖĞRENCİ NO | Çözümler |
| İMZA | |

**Do not open the exam until you are told that you may begin.**
**Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.**

1. You will have 60 minutes to answer 2 questions from a choice of 3. If you choose to answer more than 2 questions, then only your best 2 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You should write your student number on every page.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the final 10 minutes of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam 60 dakikadır. Sınavda 3 soru sorulmuştur. Bu sorulardan 2 tanesini seçerek cevaplayınız. 2'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 2 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Öğrenci numaranızı her sayfaya yazınız.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyelerden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

| 1 | 2 | 3 | TOTAL |
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ÖĞRENCİ NO

**Question 1** (Conic Sections). Consider

$$x^2 - \sqrt{3}xy + 2y^2 = 1 \qquad (1)$$

(a) [5p] Calculate the discriminant of (1).

$A = 1,\ B = -\sqrt{3},\ C = 2.$

$$\text{discriminant} = B^2 - 4AC = 3 - 8 = -5$$

(b) [5p] Is the graph of (1) an ellipse, a parabola or a hyperbola?

ellipse.

(c) [20p] Rotate the coordinate axes to change (1) into an equation that has no cross product ($xy$ or $x'y'$) term.

[HINT: First solve $\cot 2\alpha = \frac{A-C}{B}$ to find the angle of rotation $\alpha$.]

First $\cot 2\alpha = \frac{A-C}{B} = \frac{1-2}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow 2\alpha = \frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{6}.$ 5

(3 pts for $\alpha = \frac{\pi}{3}$)

Therefore

$$x = x'\cos\alpha - y'\sin\alpha = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' - y'}{2}$$

and

$$y = x'\sin\alpha + y'\cos\alpha = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' + \sqrt{3}y'}{2}.$$ 5

So $1 = x^2 - \sqrt{3}xy + 2y^2$

$$= \left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - \sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 2\left(\frac{x' + \sqrt{3}y'}{2}\right)^2$$ 5

$$= \frac{1}{4}\left[3x'^2 - \cancel{2\sqrt{3}x'y'} + y'^2 - \sqrt{3}\left(\sqrt{3}x'^2 + \cancel{2x'y'} - \sqrt{3}y'^2\right) + 2\left(x'^2 + \cancel{2\sqrt{3}x'y'} + 3y'^2\right)\right]$$

$$= \frac{1}{4}\left[2x'^2 + 10y'^2\right]$$

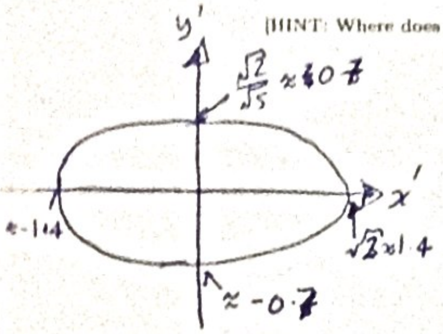
$$= \frac{1}{2}x'^2 + \frac{5}{2}y'^2.$$ 5

<u>Or</u> use formulae for $A'$, $B'$, etc.

$$x^2 - \sqrt{3}xy + 2y^2 = 1 \tag{1}$$

(d) [20p] Sketch the graph of (1). $\frac{1}{2}x'^2 + \frac{5}{2}y'^2 = 1.$

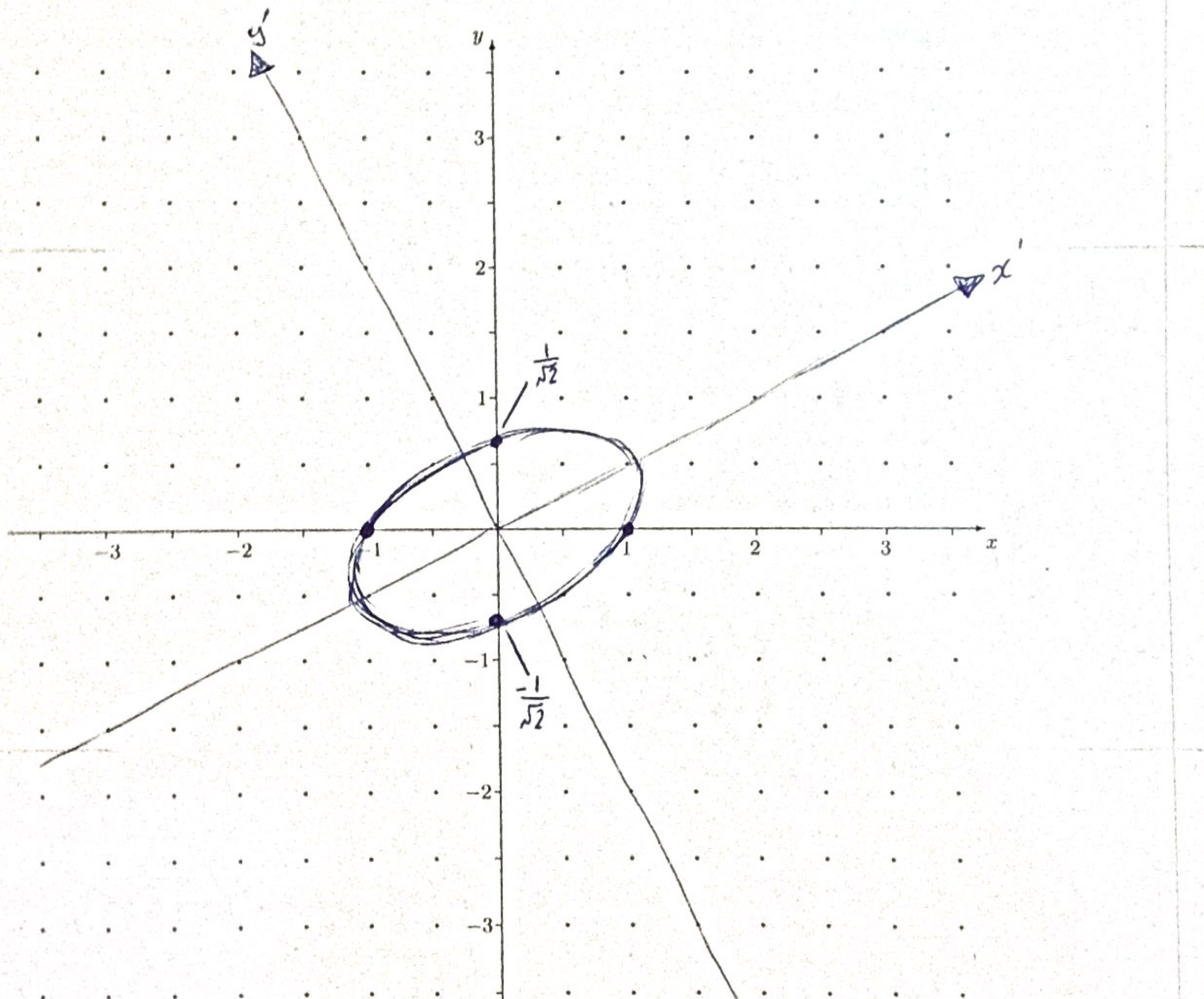
[HINT: Where does the curve cross the $x$ and $y$ axes? $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$ and $\sqrt{7} \approx 2.6$.]




$x = 0 \Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \frac{\pm 1}{\sqrt{2}} \approx \pm 0.7$

$y = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$

optional




- 5 points for $x'$ & $y'$ axis, or getting major axis in correct direction.
- 5 points for correct $x$ & $y$ intercepts. (2 for $x$-axis, 3 for $y$-axis).
- 10 points for correct shape.

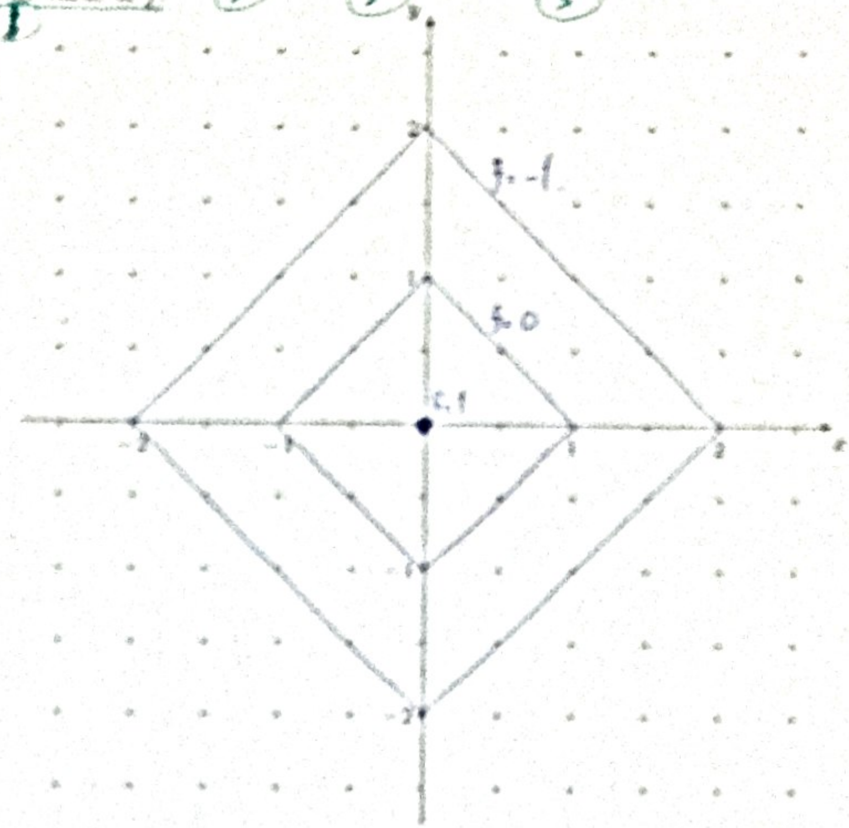
Studiennr. [illegible]

**Question 2** (Functions of Several Variables). Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 1 - |x| - |y|.$$

(a) [illegible] Plot the level curves $f(x, y) = 0$, $f(x, y) = 1$ and $f(x, y) = -1$ in $\mathbb{R}^2$. Label each level curve with the value of $f$.  5  5  5  5

[illegible]

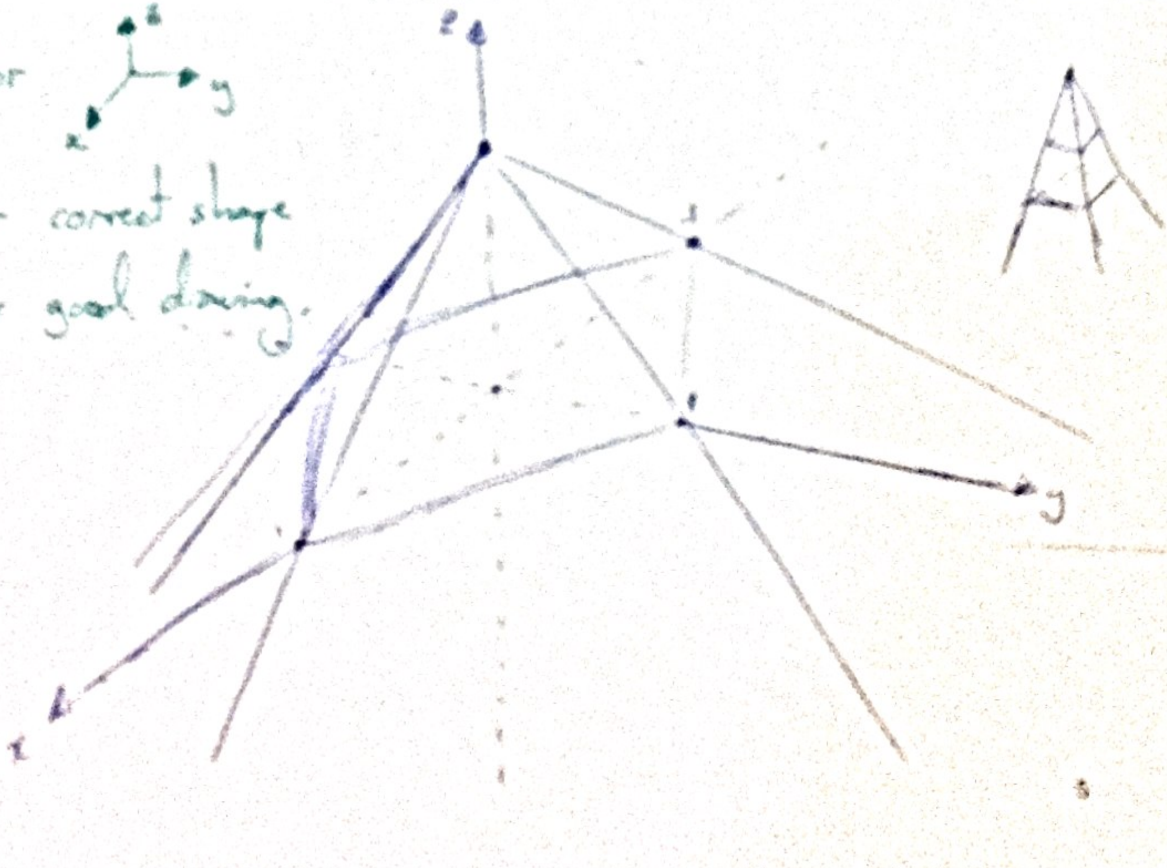



(b) [illegible] Sketch the surface $z = f(x, y)$ in $\mathbb{R}^3$.

5 for

7 for correct shape

+3 for good drawing.



Now consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = (e^{-2y} \cos 2x) + \log \sqrt{x^2 + y^2}$$

(where $\log = \log_e = \ln$ is the natural logarithm).

(c) [25p] Show that

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0.$$

1

Let $h(x,y) = e^{-2y} \cos 2x$ and $j(x,y) = \log \sqrt{x^2+y^2}$.

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Then $h_x = -2e^{-2y} \sin 2x$, $h_{xx} = -4e^{-2y} \cos 2x$,
$h_y = -2e^{-2y} \cos 2x$ and $h_{yy} = 4e^{-2y} \cos 2x$. So $h_{xx} + h_{yy} = 0$.

Moreover, $j_x = \frac{\partial}{\partial x}\left(\log\sqrt{x^2+y^2}\right) = \frac{d}{du}\left(\log u\right) \cdot \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\right)$

$$= \frac{1}{|u|} \cdot \frac{\frac{1}{2}}{\sqrt{x^2+y^2}} \cdot 2x$$

$\left(\text{since } \frac{d}{dz} z^{\frac{1}{2}} = \frac{1}{2} z^{-\frac{1}{2}}\right)$

10

$$= \frac{x}{x^2+y^2}$$

and $j_{xx} = \frac{1 \cdot (x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$.

Similarly $j_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$, so $j_{xx} + j_{yy} = 0$.

5

Therefore

$$g_{xx} + g_{yy} = 0.$$

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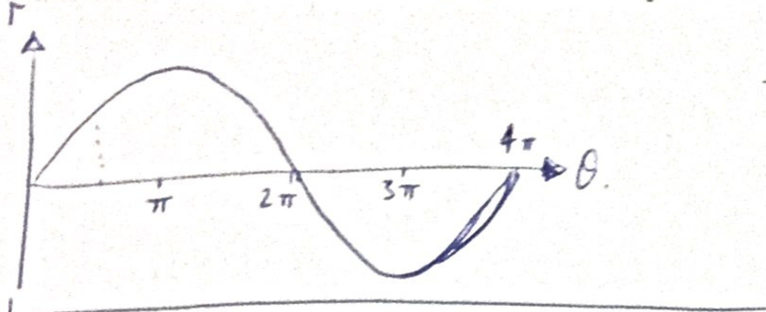
**Question 3** (Polar Coordinates).

(a) [25 pts] Graph the curve

$$r = \sin\left(\frac{\theta}{2}\right).$$

[HINT: You should consider $0 \le \theta \le 4\pi$ because $\sin\frac{\pi}{2} = 1 \neq -1 = \sin\frac{3\pi}{2}$. $\frac{1}{\sqrt{2}} \approx 0.7$.]

$r=0 \Rightarrow \theta = 0 \Rightarrow$ slope $= \tan 0 = 0$
$\theta = 2\pi \Rightarrow$ slope $= \tan 2\pi = 0$
$\theta = 4\pi \Rightarrow$ slope $= \tan 4\pi = 0$.




| $\theta$ | $r$ |
|---|---|
| $0$ | $0$ |
| $\pi/6$ | $\sin\frac{\pi}{12}$ |
| $\pi/4$ | $\sin\frac{\pi}{8}$ |
| $\pi/3$ | $\frac{1}{2}$ |
| $\pi/2$ | $\frac{1}{\sqrt{2}} \approx 0.7$ |

| $\theta$ | $r$ |
|---|---|
| $\pi$ | $1.$ |

optional.

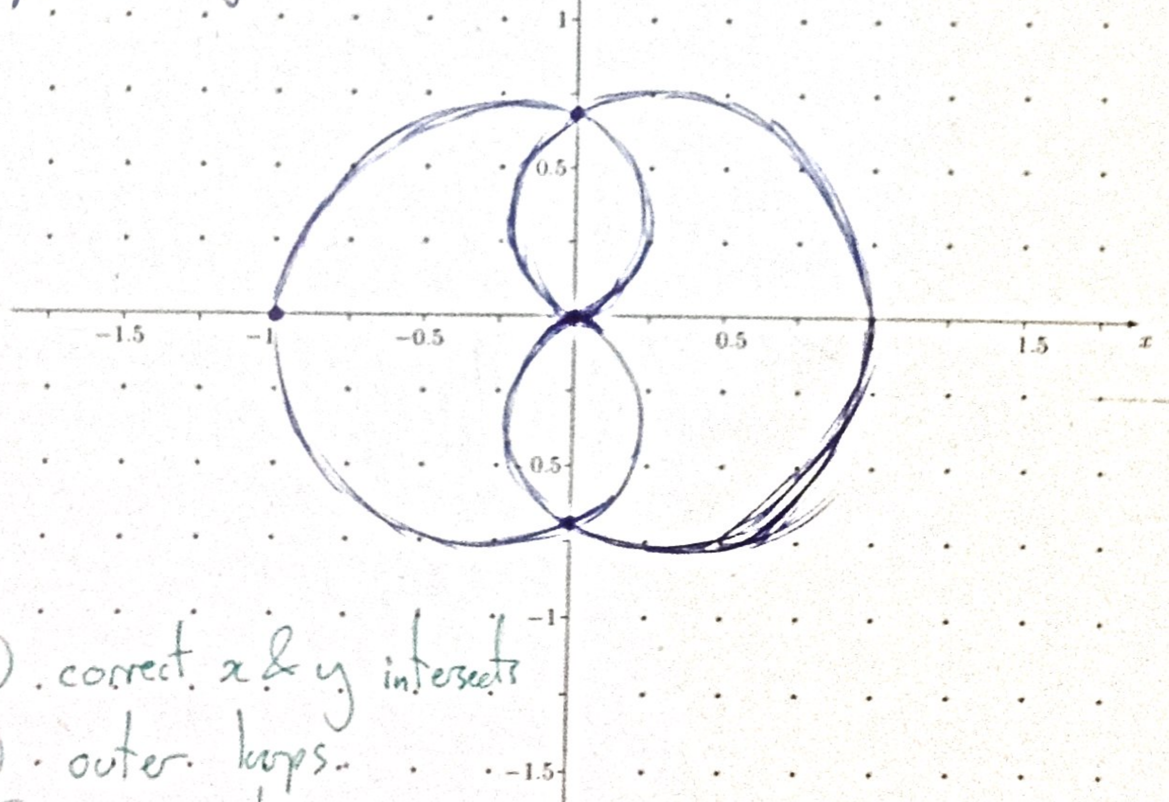
symmetry.
$(r,\theta)$ on curve $\Leftrightarrow r = \sin\left(\frac{\theta}{2}\right) \Leftrightarrow -r = \sin\left(\frac{\theta}{2}+\pi\right) \Leftrightarrow -r = \sin\left(\frac{\theta+2\pi}{2}\right)$
$\Leftrightarrow (-r,\theta)$ on curve.

So the graph is symmetrical about the origin.

Also
$(r,\theta)$ on curve $\Leftrightarrow r = \sin\left(\frac{\theta}{2}\right) \Leftrightarrow -r = \sin\left(-\frac{\theta}{2}\right) \Leftrightarrow (-r, \frac{\theta}{2})$ on curve

So the graph is symmetrical about the y-axis.

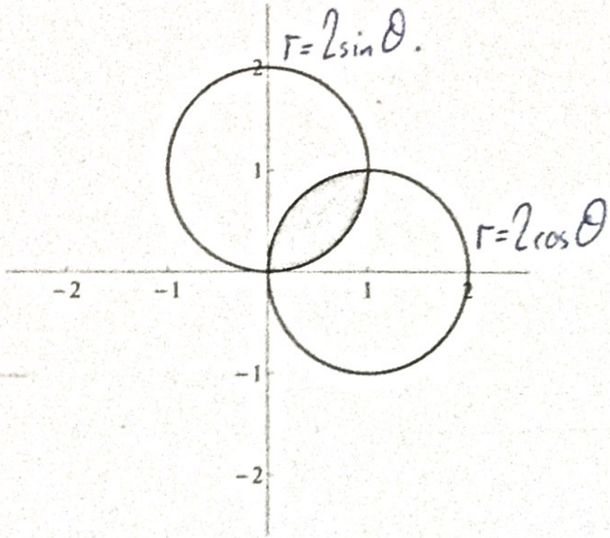
So the graph is also symmetrical about the x-axis.



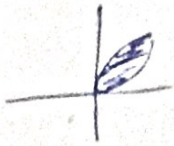
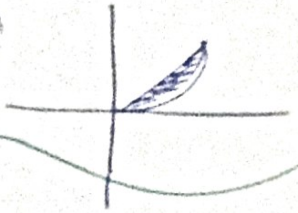

5 . correct x & y intersects
10 . outer loops.
10 . inner loops.

(+5 pts given if graph is wrong, but optional top part is included and correct)

(b) [25 pts] The region shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$ is shown below. Calculate the area of this region.




The curves intersect when $\theta = \frac{\pi}{4}$. By symmetry, 5

area of  = 2× area of  5

$$= 2\int_{\theta=0}^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi/4} (2\sin\theta)^2 \, d\theta \; 5 = 4\int_0^{\pi/4} \sin^2\theta \, d\theta$$

$$= 2\int_0^{\pi/4} 1 - \cos 2\theta \, d\theta \; 5$$

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= 2\left\{\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - (0 - 0)\right\}$$

$$\boxed{= \frac{\pi}{2} - 1} \quad 5$$