OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2014.01.09 **MAT 233 – Matematik III – Final Sınavın Çözümleri** N. Course

Soru 1 (Extrema and Saddle Points)**.**

(a) [10p] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^4 + y^4 + 4xy$. Find all the local maxima, local minima and saddle points of f.

f and its derivatives exist and are continuous on $\mathbb{R}^2$, so we must solve $f_x = 0 = f_y$. Since

$$0 = f_x(x, y) = 4x^3 + 4y \implies y = -x^3$$
$$0 = f_y(x, y) = 4y^3 + 4x = -4x^9 + 4x = 4x(1 - x^8)$$

we have that $x = -1, 0$ or 1. So the critical points are $(-1, 1)$, $(0, 0)$ and $(1, -1)$.

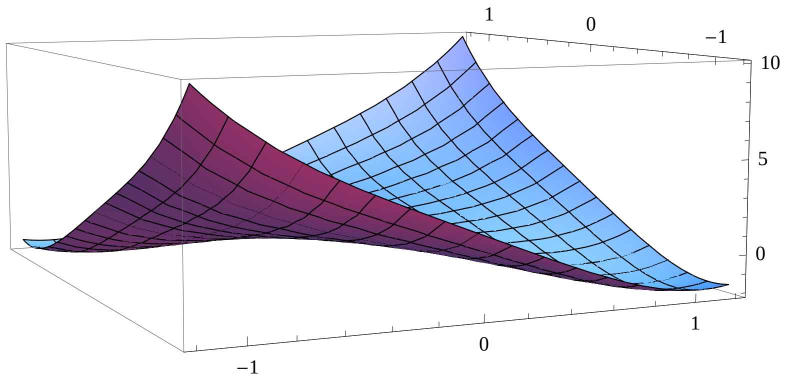
Next we calculate

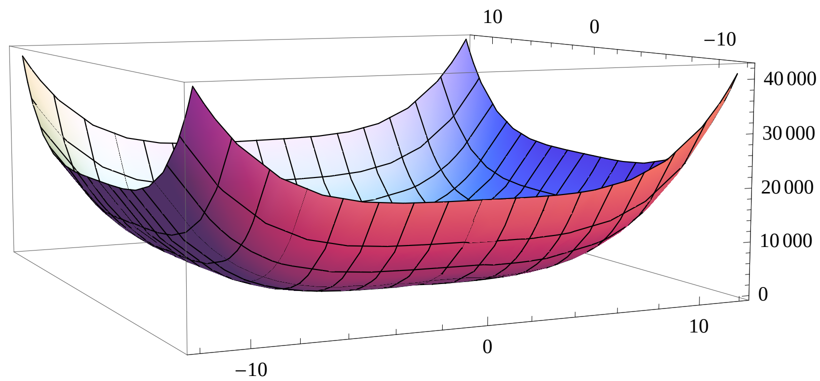
$$f_{xx} = 12x^2$$
$$f_{xy} = 4$$
$$f_{yy} = 12y^2.$$

Therefore the Hessian is $H(f) = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$.

Since $H(f)(-1, 1) = 128 > 0$ and $f_{xx}(-1, 1) = 12 > 0$, the point $(-1, 1)$ is a **local minimum**.
Similarly $H(f)(1, -1) = 128 > 0$ and $f_{xx}(1, -1) = 12 > 0$, which means that the point $(1, -1)$ is also a **local minimum**.
Finally $H(f)(0, 0) = -16 < 0$, so the point $(0, 0)$ is a **saddle point**.

If you are interested, the graph of this function looks like this:





Let $R = [1,3] \times [-\frac{\pi}{4}, \frac{\pi}{4}]$ be the closed region shown below. Let $g : R \to \mathbb{R}$ be defined by

$$g(x,y) = (4x - x^2)\cos y.$$

(b) [15p] Find the absolute maximum and absolute minimum of g on R.

(a) Interior Points: Since

$$0 = g_x = (4-2x)\cos y$$

and since $\cos y \neq 0$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$, we must have $x = 2$. Then since

$$0 = g_y = -(4x - x^2)\sin y = -(8-4)\sin y$$

we must have $y = 0$. The only interior critical point is $(2,0)$ and $g(2,0) = 4$.

(b) On the four corners: $f(1, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$, $f(3, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$ $f(3, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}$ and $f(1, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}$

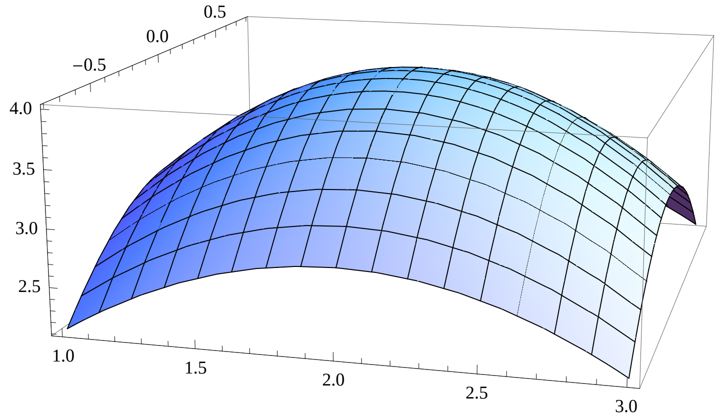
(c) On AB, $y = -\frac{\pi}{4}$. Define $h(x) = g(x, -\frac{pi}{4}) = \frac{4x-x^2}{\sqrt{2}}$. Then $0 = h'(x) = \frac{4-2x}{\sqrt{2}} \implies x = 2$ and $h(2) = \frac{4}{\sqrt{2}}$.

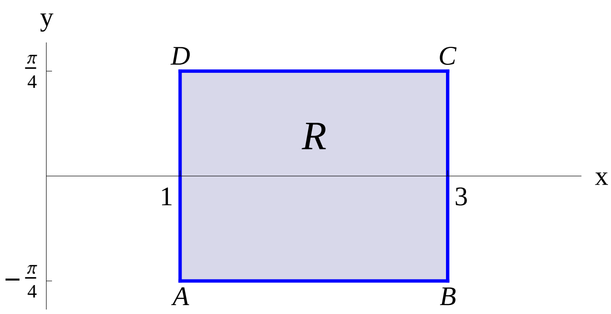
(d) On BC, $x = 3$. Define $k(y) = g(3,y) = 3\cos y$. Then $0 = k'(y) = -3\sin y \implies y = 0$ and $k(0) = 3$.

(e) On CD, $y = \frac{\pi}{4}$. Define $l(x) = g(x, \frac{pi}{4}) = \frac{4x-x^2}{\sqrt{2}} = h(x)$. We will obtain $l(2) = \frac{4}{\sqrt{2}}$ as above.

(f) On DA, $x = 1$. Define $m(y) = g(1,y) = 3\cos y = k(y)$. We will obtain $m(0) = 3$ as above.

Therefore, the absolute maximum of g on R is 4, and the absolute minimum of g on R is $\frac{3}{\sqrt{2}}$.





Soru 2 (Partial Derivatives, The Chain Rule and Directional Derivatives).

(a) [5p] Suppose that $g(x,y) = xe^{\frac{y^2}{2}}$. Calculate $\frac{\partial^5 g}{\partial x^2 \partial y^3}$.

Since g and all its partial derivatives are continuous, we can change the order of differentiation. Hence

$$\frac{\partial^5 g}{\partial x^2 \partial y^3} = \frac{\partial^5 g}{\partial y^3 \partial x^2} = \frac{\partial^5}{\partial y^3 \partial x}\left[e^{\frac{y^2}{2}}\right] = \frac{\partial^5}{\partial y^3}[0] = 0.$$

(b) [10p] Suppose that $w = xy + yz + xz$ where $x = u + v$, $y = u - v$ and $z = uv$.

<u>Use the Chain Rule</u> to calculate

$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(\frac{1}{2},1)} \quad \text{and} \quad \left.\frac{\partial w}{\partial v}\right|_{(u,v)=(\frac{1}{2},1)}.$$

Since

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \\ &= (y+z)(1) + (x+z)(1) + (x+y)(v) \\ &= (u - v + uv) + (u + v + uv) + 2uv \\ &= 2u + 4uv\end{aligned}$$

and

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \\ &= (y+z)(1) + (x+z)(-1) + (x+y)(u) \\ &= (u - v + uv) - (u + v + uv) + 2u^2 \\ &= -2v + 2u^2,\end{aligned}$$

we have that

$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(\frac{1}{2},1)} = 3 \quad \text{and} \quad \left.\frac{\partial w}{\partial v}\right|_{(u,v)=(\frac{1}{2},1)} = -\frac{3}{2}.$$

-1 point if d is used anywhere instead of ∂.

(c) [10p] Let $f(x,y) = xy + yz + xz$, $P_0 = (1,-1,2)$ and $v = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$. Calculate the derivative of f at the point P_0 in the direction $\mathbf{v}$.

[HINT: $\mathbf{v}$ is not a unit vector.]

The direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{9+36+4}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k} \quad \boxed{3}$$

and the gradient of f at P_0 is

$$\nabla f|_{P_0} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}|_{P_0} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}|_{P_0} = \mathbf{i} + 3\mathbf{j}. \quad \boxed{3}$$

Therefore, the derivative of f at the point P_0 in the direction $\mathbf{v}$ is

$$(D_{\mathbf{u}}f)_{P_0} = \nabla f|_{P_0} \bullet \mathbf{u} = (\mathbf{i} + 3\mathbf{j}) \bullet \left(\frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}\right) = 3. \quad \boxed{4}$$

Soru 3 (Spherical Polar Coordinates)**.** Let $D \subseteq \mathbb{R}^3$ be the region enclosed by $\rho = 2\sin\phi$. Define a function $F : D \to \mathbb{R}$ by $F(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}$

[25p] Calculate the average value of F on D.

[HINT: $\int_0^{\pi/2}\cos\zeta\;d\zeta = 1$, $\int_0^{\pi/2}\sin\zeta\;d\zeta = 1$, $\int_0^{\pi/2}\cos^2\zeta\;d\zeta = \frac{\pi}{4}$, $\int_0^{\pi/2}\sin^2\zeta\;d\zeta = \frac{\pi}{4}$, $\int_0^{\pi/2}\cos^3\zeta\;d\zeta = \frac{2}{3}$, $\int_0^{\pi/2}\sin^3\zeta\;d\zeta = \frac{2}{3}$, $\int_0^{\pi/2}\cos^4\zeta\;d\zeta = \frac{3\pi}{16}$, $\int_0^{\pi/2}\sin^4\zeta\;d\zeta = \frac{3\pi}{16}$, $\int_0^{\pi/2}\cos^5\zeta\;d\zeta = \frac{8}{15}$, $\int_0^{\pi/2}\sin^5\zeta\;d\zeta = \frac{8}{15}$, $\int_0^{\pi/2}\cos^6\zeta\;d\zeta = \frac{5\pi}{32}$. $\int_0^{\pi/2}\sin^6\zeta\;d\zeta = \frac{5\pi}{32}$.]

First, the volume of D is

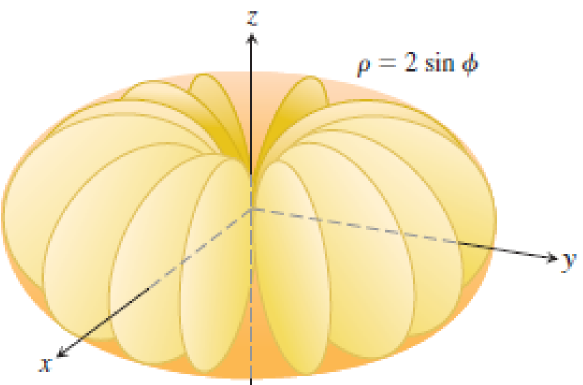
$$\begin{aligned}
V = \iiint_D dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2\sin\phi\;d\rho d\phi d\theta \quad \boxed{6}\\
&= 8\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2\sin\phi} \rho^2\sin\phi\;d\rho d\phi d\theta = \frac{64}{3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^4\phi\;d\phi d\theta\\
&= \frac{64}{3}\int_0^{\frac{1}{2}\pi}\frac{3\pi}{16}\;d\theta \qquad \text{(by the hint)}\\
&= 4\pi\int_0^{\frac{1}{2}\pi} d\theta = 2\pi^2. \quad \boxed{6}
\end{aligned}$$

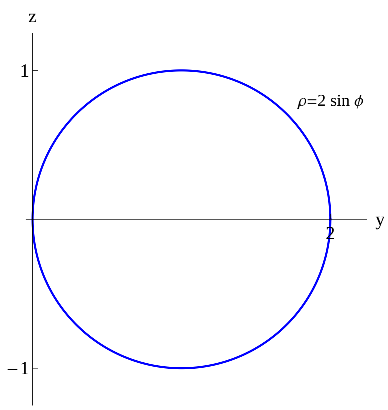
Next, since $x^2+y^2 = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\sin^2\phi$, we have that $F(x,y,z) = \frac{1}{\rho\sin\phi}$. Hence

$$\begin{aligned}
\iiint_D F(x,y,z)\;dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{2\sin\phi}\left(\frac{1}{\rho\sin\phi}\right)\rho^2\sin\phi\;d\rho d\phi d\theta\\
&= 8\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2\sin\phi}\rho\;d\rho d\phi d\theta \quad \boxed{6}\\
&= 16\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^2\phi\;d\phi d\theta\\
&= 16\int_0^{\frac{1}{2}\pi}\frac{\pi}{4}\;d\theta \qquad \text{(by the hint)}\\
&= 4\pi\int_0^{\frac{1}{2}\pi}d\theta = 2\pi^2 \quad \boxed{6}
\end{aligned}$$

Therefore, the average value of F on D is

$$\frac{1}{V}\iiint_D F(x,y,z)\;dV = \frac{2\pi^2}{2\pi^2} = 1. \quad \boxed{1}$$





Soru 4 (Lagrange Multipliers)**.** Consider the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(a) [23p] Of all the rectangles that can fit inside the ellipse, **use a Lagrange Multiplier** to find the width and the height of the rectangle with the **biggest area**.

(b) [2p] What is the area of this rectangle?

(a) Suppose that the rectangle has vertices at $(\pm x, \pm y)$, where $x > 0$ and $y > 0$. Then the area of the rectangle is

$$f(x, y) = (2x)(2y) = 4xy.$$

Define a function g by

$$g(x, y) = \frac{x^2}{16} + \frac{y^2}{9} - 1.$$

We need to find the maximum of f, subject to the constraint $g = 0$. So we need to find numbers x, y, λ such that $\nabla f = \lambda \nabla g$ and $g(x, y) = 0$.

Since

$$4y\mathbf{i} + 4x\mathbf{j} = \nabla f = \lambda \nabla g = \frac{\lambda x}{8}\mathbf{i} + \frac{2\lambda y}{9}\mathbf{j}$$

we must have

$$\begin{cases} 4y = \frac{\lambda}{8}x \\ 4x = \frac{2\lambda}{9}y \end{cases}$$

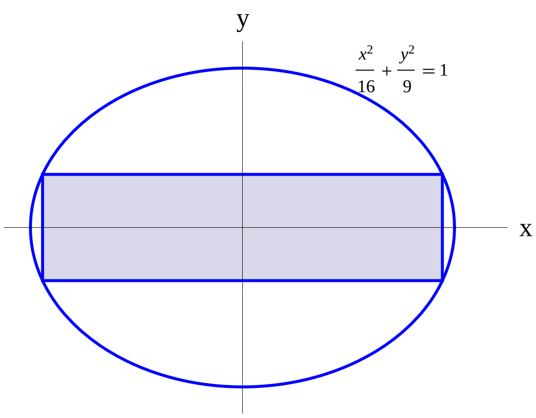
which implies that $\lambda = \frac{32y}{x}$ and $4x = \left(\frac{2y}{9}\right)\left(\frac{32y}{x}\right) = \frac{64y^2}{9x}$. So $y = \pm\frac{3}{4}x$. Therefore

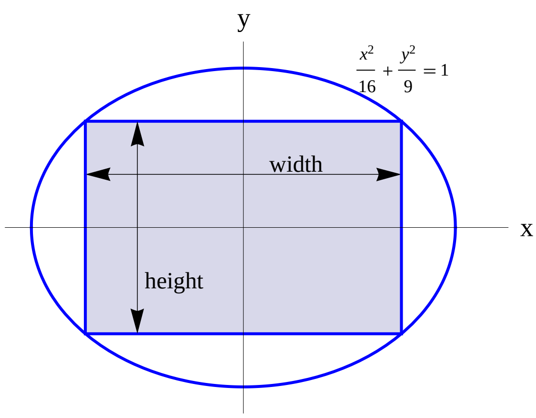
$$0 = g(x, y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = \frac{x^2}{16} + \frac{x^2}{16} - 1 \qquad \Longrightarrow \qquad x^2 = 8.$$

Since we want $x > 0$, we have $x = 2\sqrt{2}$ and $y = \frac{3}{4}x = \frac{3\sqrt{2}}{2}$.

Therefore, the width of the rectangle should be $2x = \underline{4\sqrt{2}}$ and the height of the rectangle should be $2y = \underline{3\sqrt{2}}$.

(b) The area of rectangle is $(2x)(2y) = 24$.





Soru 5 (Substitutions in Multiple Integrals)**.** Let R be the region bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > 0$ and $b > 0$ are constants.

[25p] Use the transformation

$$x = ar\cos\theta \quad \text{and} \quad y = br\sin\theta,$$

to calculate

$$\iint_R \left(x^2 + y^2\right) dxdy.$$

And an easy question to finish with: First we calculate the Jacobian of this transformation,

$$J(r,\theta) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr.$$

The region R is given by $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Therefore

$$\begin{aligned}
\iint_R \left(x^2 + y^2\right) dxdy &= \int_0^{2\pi} \int_0^1 r^2(a^2\cos^2\theta + b^2\sin^2\theta)\,|J|\,drd\theta \\
&= \int_0^{2\pi} \int_0^1 abr^3(a^2\cos^2\theta + b^2\sin^2\theta)drd\theta \\
&= \frac{ab}{4}\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)d\theta \\
&= \frac{ab}{4}\left[\frac{1}{2}a^2\theta + \frac{1}{4}a^2\sin 2\theta + \frac{1}{2}b^2\theta - \frac{1}{4}b^2\sin 2\theta\right]_0^{2\pi} \\
&= \frac{1}{4}ab\pi(a^2 + b^2).
\end{aligned}$$