2013.11.13 **MAT 233 – Matematik III – Ara Sınavın Çözümleri** N. Course

Soru 1 (Conic Sections)**.** Consider

$$3x^2 + 4\sqrt{3}xy - y^2 = 7. \tag{1}$$

(a) [5p] The graph of (**??**) is

☐ an ellipse, ☐ a parabola, ☑ a hyperbola.

optional The discriminant is

$$B^2 - 4AC = 16 \times 3 - 4 \times 3 \times (-1) = 48 + 12 = 60 > 0.$$

Therefore the conic section is a hyperbola.

(b) [25p] Rotate the coordinate axes to change (**??**) into an equation that has no cross product (xy or $x'y'$) term.

[HINT: First solve $\cot 2\alpha = \frac{A-C}{B}$ to find the angle of rotation α.]

Since $\cot 2\alpha = \frac{A-C}{B} = \frac{3-(-1)}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \implies 2\alpha = \frac{\pi}{3}$, we have that $\alpha = \frac{\pi}{6}$ 5. Therefore

$$x = x'\cos\alpha - y'\sin\alpha = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} = \frac{\sqrt{3}x' - y'}{2} \quad \boxed{4}$$

and

$$y = x'\sin\alpha + y'\cos\alpha = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} = \frac{x' + \sqrt{3}y'}{2}. \quad \boxed{4}$$

It follows that

$$\begin{aligned}
7 &= 3x^2 + 4\sqrt{3}xy - y^2 \\
&= 3\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 + 4\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) - \left(\frac{x' + \sqrt{3}y'}{2}\right)^2 \quad \boxed{4} \\
&= \frac{1}{4}\left[3\left(\sqrt{3}x' - y'\right)^2 + 4\sqrt{3}\left(\sqrt{3}x' - y'\right)\left(x' + \sqrt{3}y'\right) - \left(x' + \sqrt{3}y'\right)^2\right] \\
&= \frac{1}{4}\left[9x'^2 - 6\sqrt{3}x'y' + 3y'^2 + 4\sqrt{3}\left(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2\right) - x'^2 - 2\sqrt{3}x'y' - 3y'^2\right] \quad \boxed{4} \\
&= \frac{1}{4}\left[20x'^2 - 12y'^2\right] \\
&= 5x'^2 - 3y'^2. \quad \boxed{4}
\end{aligned}$$

Therefore the answer is $5x'^2 - 3y'^2 = 7$.

Alternately, use the formulae for A', B', etc

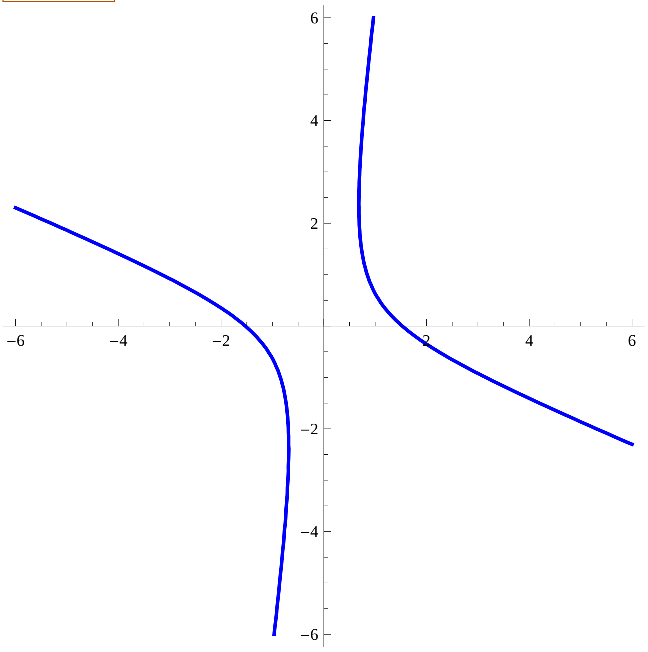
(c) [20p] Sketch the graph of (**??**).

[HINT: Where does the curve cross the x and y axes? $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$ and $\sqrt{7} \approx 2.6$.]

$$5x'^2 - 3y'^2 = 7$$

$y = 0 \implies 7 = 3x^2 \implies x^2 = \frac{7}{3} \implies x \approx \pm\frac{2.6}{1.7} \approx \pm 1.5$ and $y = 0 \implies -y^2 = 7$ which has no solutions. The graph crosses the x-axis at $\approx \pm 1.5$ and does not cross the y-axis.

optional

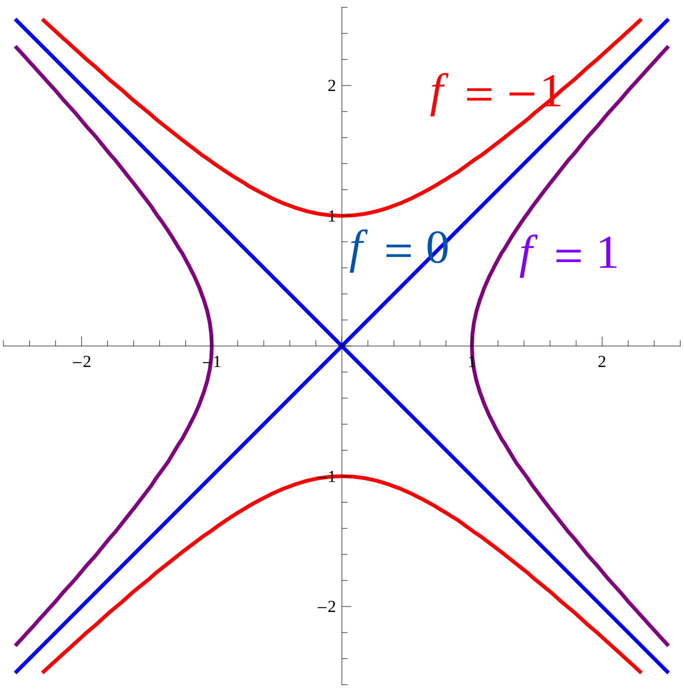


5 points for x and y intercepts | 5 points for correct angle of rotation | 10 pts for shape

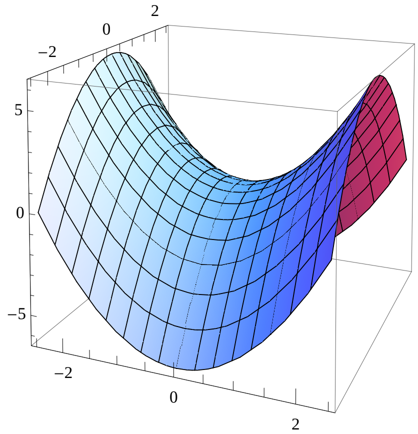
Soru 2 (Functions of Several Variables)**.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = x^2 - y^2.$$

(a) [10p] Plot the level curves $f(x, y) = 0$, $f(x, y) = 1$ and $f(x, y) = -1$ in $\mathbb{R}^2$. Label each level curve with the value of f.



(b) [15p] Sketch the surface $z = f(x, y)$ in $\mathbb{R}^3$.



Now suppose that

$$w(x,y) = \tan^{-1}\left(\frac{x}{y}\right), \qquad x(t) = \cos t, \qquad y(t) = \sin t.$$

(c) [5p] Calculate $\frac{\partial w}{\partial x}$.
[HINT: $\frac{d}{dz}\tan^{-1} z = \frac{1}{z^2+1}$]

Since $\frac{d}{dz}\tan^{-1} z = \frac{1}{z^2+1}$ we have that

$$\frac{\partial w}{\partial x} = \frac{\frac{1}{y}}{\left(\frac{x}{y}\right)^2 + 1} = \frac{y}{x^2+y^2}$$

(d) [5p] Calculate $\frac{\partial}{\partial y}\left(\frac{x}{y}\right)$.

$$\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2}$$

(e) [5p] Calculate $\frac{\partial w}{\partial y}$.

$$\frac{\partial w}{\partial y} = \frac{-\frac{x}{y^2}}{\left(\frac{x}{y}\right)^2 + 1} = \frac{-x}{x^2+y^2}$$

(f) [10p] <u>Use the Chain Rule</u> to calculate

$$\left.\frac{dw}{dt}\right|_{t=\pi/6}$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{y}{x^2+y^2}\right)(-\sin t) + \left(\frac{-x}{x^2+y^2}\right)\cos t \\ &= -\sin^2 t - \cos^2 t \\ &= -1\end{aligned}$$

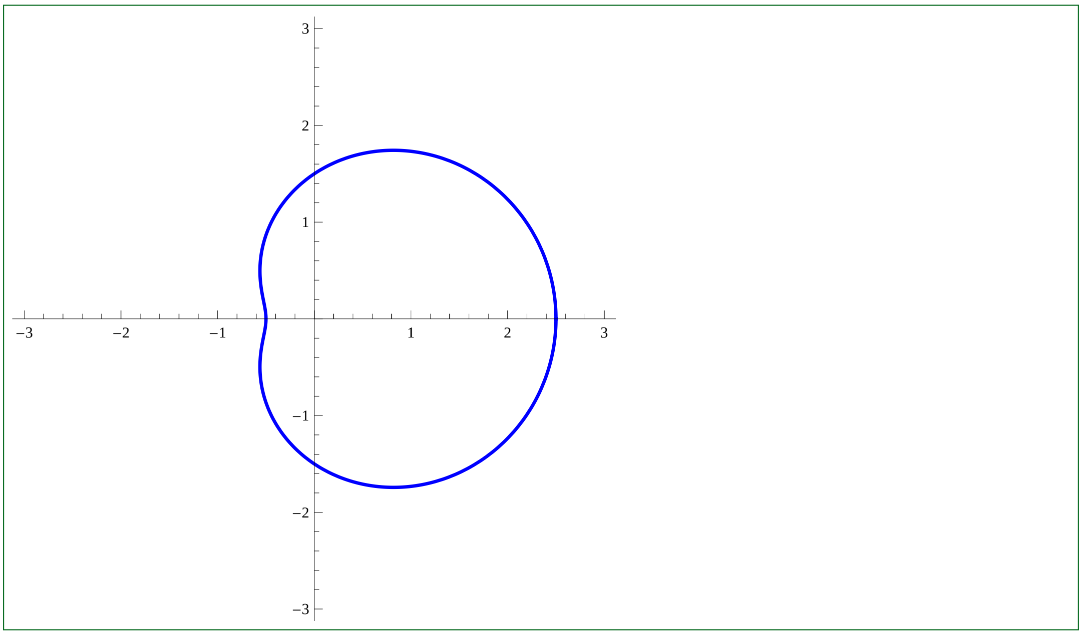
2 points. 0 points if d confused with ∂

since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Therefore $\left.\frac{dw}{dt}\right|_{t=\pi/6} = -1$

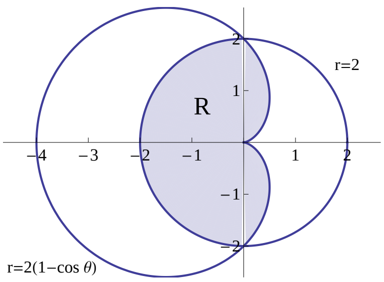
Soru 3 (Polar Coordinates)**.**

(a) [25p] Graph the curve

$$r = \frac{3}{2} + \cos\theta.$$



Let R be the region enclosed by both the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$, as shown below.



(b) [25p] Calculate the area of R.

The curves intersect when $\theta = \pm\frac{\pi}{2}$. Therefore the area of R is

$$\begin{aligned}
A &= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[2(1-\cos\theta)\right]^2 \, d\theta + \frac{1}{2}(\text{area of the circle}) \\
&= 4\int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \cos^2\theta \, d\theta + \frac{1}{2}\pi 2^2 \\
&= 4\int_0^{\frac{\pi}{2}} 1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta) \, d\theta + 2\pi \\
&= \int_0^{\frac{\pi}{2}} 6 - 8\cos\theta + 2\cos 2\theta \, d\theta + 2\pi \\
&= \Big[6\theta - 8\sin\theta + \sin 2\theta\Big]_0^{\frac{\pi}{2}} + 2\pi \\
&= 5\pi - 8
\end{aligned}$$