2015.01.06 **MAT233 Matematik III – Final Sınavın Çözümleri** N. Course

Soru 1 (Partial Derivatives, The Chain Rule and Directional Derivatives)**.** Suppose that $w = x^2 + \frac{y}{x}$ where $x = u - 2v + 1$ and $y = 2u + v - 2$.

(a) [12p] **Use the Chain Rule** to calculate

$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(0,0)} \qquad \text{and} \qquad \left.\frac{\partial w}{\partial v}\right|_{(u,v)=(0,0)}.$$

Notice first that if $(u, v) = (0, 0)$ then $(x, y) = (1, -2)$.
Since

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} \\ &= \left(2x - \frac{y}{x^2}\right)(1) + \left(\frac{1}{x}\right)(2) \\ &= 2x + \frac{2}{x} - \frac{y}{x^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} \\ &= \left(2x - \frac{y}{x^2}\right)(-2) + \left(\frac{1}{x}\right)(1) \\ &= -4x + \frac{1}{x} + \frac{2y}{x^2},\end{aligned}$$

we have that

$$\left.\frac{\partial w}{\partial u}\right|_{(u,v)=(0,0)} = 2 + 2 - \frac{-2}{1} = 2 \qquad \text{and} \qquad \left.\frac{\partial w}{\partial v}\right|_{(u,v)=(0,0)} = -4 + 1 + \frac{-4}{1} = -7.$$

Let $f(x, y) = x^2 + 2y^2 - 3z^2$, $P_0 = (1, 1, 1)$ and $v = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(b) [13p] Calculate the derivative of f at the point P_0 in the direction $\mathbf{v}$.
[HINT: $\mathbf{v}$ is not a unit vector.]

The direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{i} + \mathbf{j} + 1\mathbf{k}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

and the gradient of f at P_0 is

$$\nabla f|_{P_0} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}|_{P_0} = 2x\mathbf{i} + 4y\mathbf{j} - 6z\mathbf{k}|_{P_0} = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}.$$

Therefore, the derivative of f at the point P_0 in the direction $\mathbf{v}$ is

$$(D_{\mathbf{u}}f)_{P_0} = \nabla f|_{P_0} \bullet \mathbf{u} = (2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) \bullet \left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) = \frac{2+4-6}{\sqrt{3}} = 0.$$

Soru 2 (Lagrange Multipliers).

English

I want to make a cuboid box of width x metres, depth y metres and height z metres. The volume of my box must be 60m^3.

The top of the box must be left open. The front and the base of the box must be made of metal. The other three sides must be made of plastic.

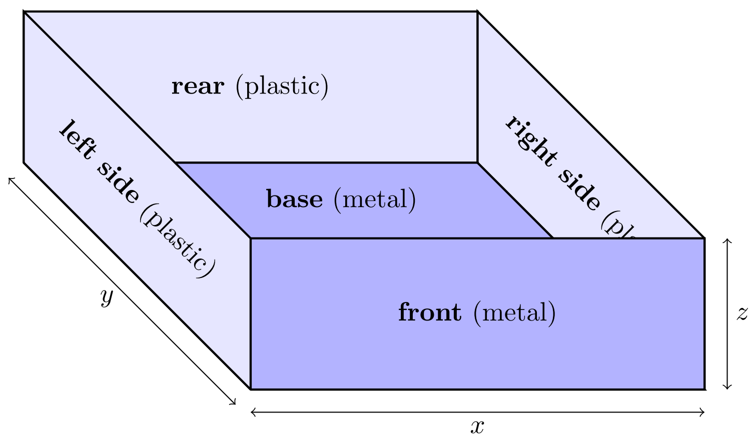
Metal costs 5 Turkish Lira per square metre. Plastic costs 1 Turkish Lira/m^2. Other than buying the materials, there are no other costs involved in making the box.

Türkçe

Genişliği x metre, derinliği y metre ve yüksekliği z metre olan küboid bir kutu yapmak istiyorum. Kutunun hacmi 60m^3 olmalı.

Kutunun üstü açık bırakılmalı. Kutunun ön yüzü ve tabanı metalden yapılmalı. Diğer üç yüzü plastikten yapılmalı.

Metalin metrekaresi 5 Türk Lirasina mal olmakta. Plastiğin maliyeti 1 Türk Lirasi/m^2 dir. Malzemelerin dışında kutuyu yapmak için başka bir maliyet yoktur.



I want to make the box as cheaply as possible.

[25p] **Use a Lagrange Multiplier** to find which dimensions (x, y and z) I should use.

Kutuyu mümkün olan en ucuz şekilde yapmak istiyorum.

[25p] **Lagrange Çarpanı'nı** kullanarak hangi boyutları (x, y ve z için) kullanmam gerektiğini bulunuz.

Let $g(x, y, z) = 60 - xyz$. The base of the box has volume xy so costs $5xy$ liras to make. The other 4 sides are similar: We have a total cost of

$$f(x, y, z) = 5xy + 6xz + 2yz$$

Turkish Liras.

I want to find the minimum of f subject to the constraint $g = 0$. So I need to find numbers $x, y, z, \lambda \in \mathbb{R}$ such that $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 0$.

Since

$$(5y + 6z)\mathbf{i} + (5x + 2z)\mathbf{j} + (6x + 2y)\mathbf{k} = \nabla f = \lambda \nabla g = \lambda yz\mathbf{i} + \lambda xz\mathbf{j} + \lambda xy\mathbf{k}$$

we get 3 equations

$$\begin{aligned} 5y + 6z &= \lambda yz \\ 5x + 2z &= \lambda xz \\ 6x + 2y &= \lambda xy \end{aligned}$$

Rearranging we have $\lambda = \frac{5}{z} + \frac{6}{y} = \frac{5}{z} + \frac{2}{x} = \frac{6}{y} + \frac{2}{x}$ which implies that $y = 3x$ and $z = \frac{5}{2}x$. So

$$0 = g(x, y, z) = g(x, 3x, \tfrac{5}{2}x) = 60 - \frac{15}{2}x^3.$$

Rearranging gives $x^3 = 8$. Therefore $x = 2$, $y = 6$ and $z = 5$.

Soru 3 (Substitutions in Multiple Integrals)**.** Let R be the region bounded by the lines $y = -\frac{3}{2}x + 1$, $y = -\frac{3}{2}x + 3$, $y = -\frac{x}{4}$ and $y = -\frac{x}{4} + 1$.

[25p] Use the transformation $u = 3x + 2y$ and $v = x + 4y$, to calculate

$\iint_R \left(3x^2 + 14xy + 8y^2\right) dxdy$.

[HINT: $uv =$?]

Rearranging, we have

$$x = \frac{1}{5}(2u - v) \qquad \text{and} \qquad y = \frac{1}{10}(3v - u).$$

First we calculate the Jacobian of this transformation,

$$J(u, v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}.$$

The region R is given by $2 \leq u \leq 6$ and $0 \leq v \leq 4$. Therefore

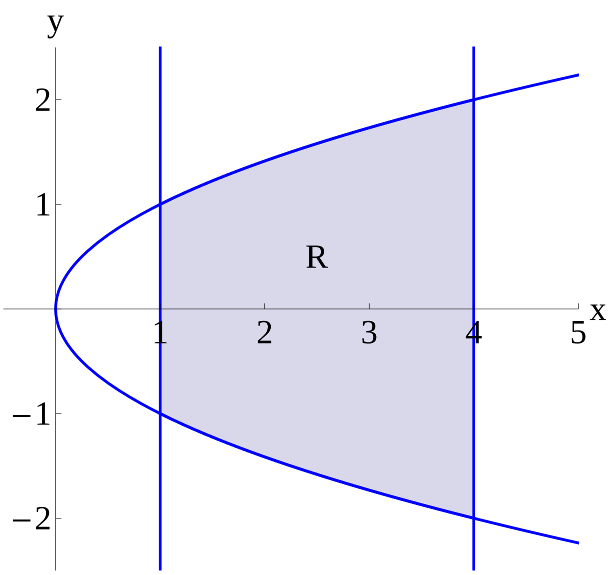
$$\begin{aligned} \iint_R \left(3x^2 + 14xy + 8y^2\right) dxdy &= \int_0^4 \int_2^6 uv\,|J|\,dudv \\ &= \frac{1}{10} \int_0^4 \int_2^6 uv\ dudv \\ &= \frac{1}{10} \int_0^4 \left[\frac{1}{2}u^2 v\right]_2^6 dv \\ &= \frac{1}{10} \int_0^4 16v\ dv \\ &= \frac{1}{10} \left[8v^2\right]_0^4 \\ &= \frac{64}{5}. \end{aligned}$$

Soru 4 (Double Integrals)**.** Let $g : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x, y) = \tfrac{3}{2} e^{\left(\frac{y}{\sqrt{x}}\right)}.$$

Let $R \subseteq \mathbb{R}^2$ be the region bounded by the curves $x = 1$, $x = 4$ and $x = y^2$.

(a) [5p] Sketch R.



(b) [20p] Calculate

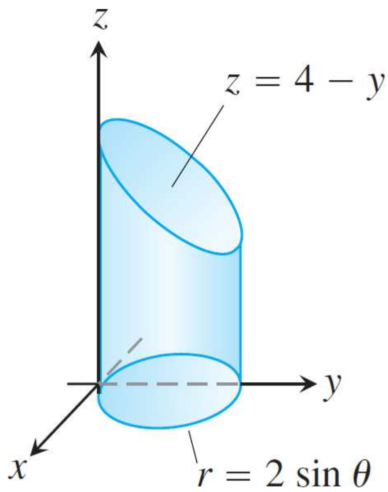
$$\iint_R g \; dA.$$

Clearly $1 \le x \le 4$ and $-\sqrt{x} \le y \le \sqrt{x}$. Therefore

$$\begin{aligned}
\iint_R g \; dA &= \int_1^4 \int_{-\sqrt{x}}^{\sqrt{x}} \tfrac{3}{2} e^{\left(\frac{y}{\sqrt{x}}\right)} dy dx \\
&= \int_1^4 \left[\frac{3}{2} \sqrt{x} e^{\left(\frac{y}{\sqrt{x}}\right)} \right]_{-\sqrt{x}}^{\sqrt{x}} dx \\
&= \int_1^4 \frac{3}{2} \sqrt{x} (e^1 - e^{-1}) dx \\
&= (e^1 - e^{-1}) \left[x^{\frac{3}{2}} \right]_1^4 \\
&= 7(e^1 - e^{-1})
\end{aligned}$$

Soru 5 (Cylindrical Polar Coordinates)**.** Let $D \subseteq \mathbb{R}^3$ be the region shown below.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$



[25p] **Use a triple integral** to calculate the volume of D.

[HINT: $\int_0^\pi \sin^2\theta\ d\theta = \frac{1}{2}\pi$ and $\int_0^\pi \sin^4\theta\ d\theta = \frac{3}{8}\pi$.]

The volume of D is

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_0^\pi \int_0^{2\sin\theta} \int_0^{4-r\sin\theta} r\ dz dr d\theta \\ &= \int_0^\pi \int_0^{2\sin\theta} 4r - r^2\sin\theta\ dr d\theta \\ &= \int_0^\pi \left[2r^2 - \frac{1}{3}r^3\sin\theta\right]_0^{2\sin\theta} d\theta \\ &= \int_0^\pi 8\sin^2\theta - \frac{8}{3}\sin^4\theta\ d\theta \\ &= 8\int_0^\pi \sin^2\theta\ d\theta - \frac{8}{3}\int_0^\pi \sin^4\theta\ d\theta \\ &= 4\pi - \pi \\ &= 3\pi. \end{aligned}$$