2014.11.20 **MAT233 Matematik III – Ara Sınavın Çözümleri** N. Course

Soru 1 (Shifting Conic Sections)**.** Consider

$$x^2 - 4y^2 + 2x + 8y - 7 = 0 \tag{1}$$

(a) [5p] The graph of (1) is

☐ an ellipse, ☐ a parabola, ☑ a hyperbola.

The discriminant is

$$B^2 - 4AC = 0 - 4 \times 1 \times -4 = 0 + 16 > 0.$$

Therefore the conic section is a hyperbola.

(b) [5p] Rearrange (1) to the form

$$\frac{(x-x_0)^2}{A} + \frac{(y-y_0)^2}{B} = 1.$$

First we complete the square in x and y to put the equation in standard form:

$$\begin{aligned} x^2 - 4y^2 + 2x + 8y &= 7 \\ (x^2 + 2x + 1) - 4(y^2 - 2y + 1) &= 7 + 1 - 4 \\ \frac{(x+1)^2}{4} - (y-1)^2 &= 1 \end{aligned}$$

(c) [5p] Find the centre of (1).

$$\frac{(x+1)^2}{4} - (y-1)^2 = 1$$

This is the standard equation for a hyperbola ($\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$), but with X replaced by $x+1$ and with Y replaced with $y-1$. Therefore the centre is $(-1, 1)$.

(d) [5p] Find the asymptotes of (1).

We find the asymptotes by replacing the 1 with a 0:

$$\begin{aligned} \frac{(x+1)^2}{4} - (y-1)^2 &= 0 \\ (y-1)^2 &= \frac{(x+1)^2}{4} \\ y - 1 &= \pm\frac{1}{2}(x+1) \\ y &= 1 \pm \frac{1}{2}(x+1) \\ y &= \frac{1}{2} - \frac{1}{2}x \quad \text{or} \quad y = \frac{3}{2} + \frac{1}{2}x \end{aligned}$$

(e) [5p] Find the focus/foci of (1).

Clearly we have $a = 2$ and $b = 1$. So $c^2 = a^2 + b^2 = 5$. The equation $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$ has foci $(\pm\sqrt{5}, 0)$. Therefore our shifted equation has foci $(-1 \pm \sqrt{5}, 1)$.

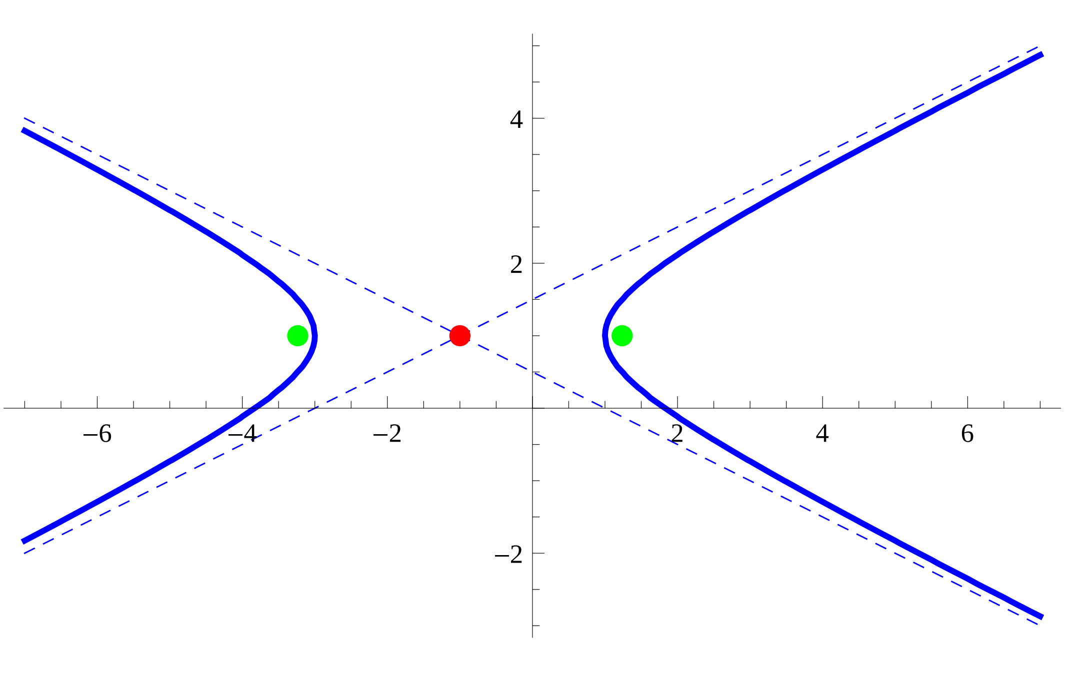
(f) [5p] Find the vertex/vertices of (1).

The equation $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$ has vertices $(\pm a, 0) = (\pm 2, 0)$. Therefore our shifted equation has vertices $(-3, 1)$ and $(1, 1)$

$$x^2 - 4y^2 + 2x + 8y - 7 = 0 \tag{1}$$

(g) [20p] Sketch the graph of (1). Include the centre, focus/foci and asymptotes in your sketch.

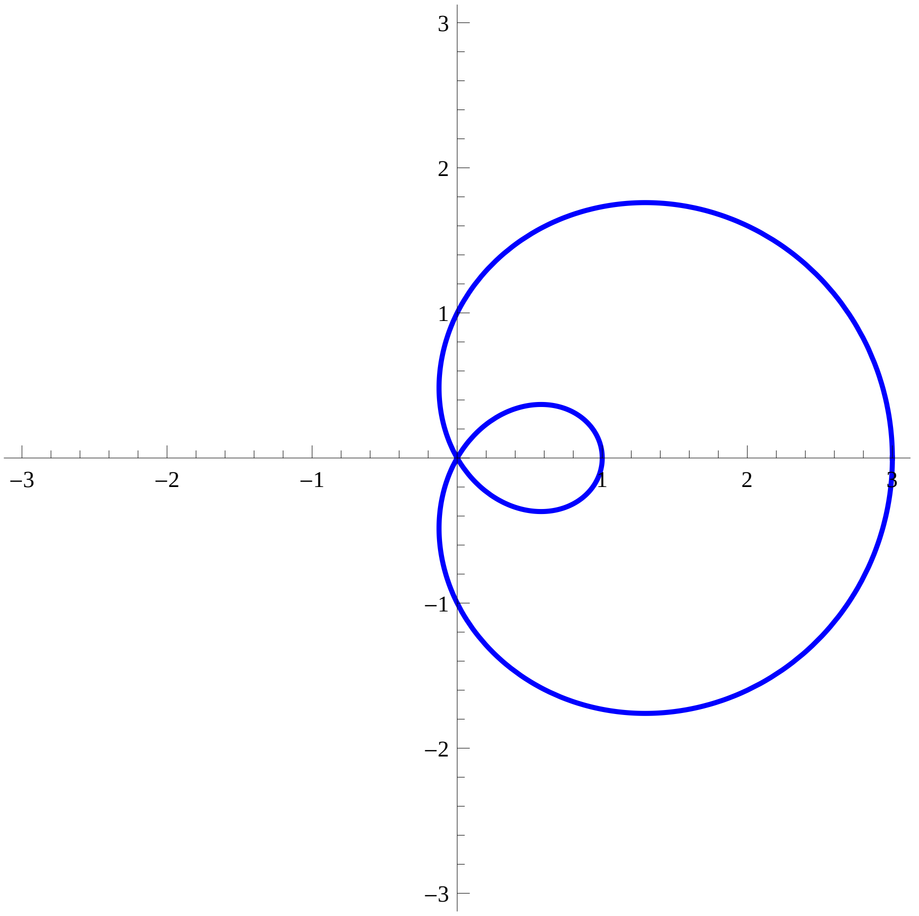
[HINT: $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$ and $\sqrt{7} \approx 2.6$.]



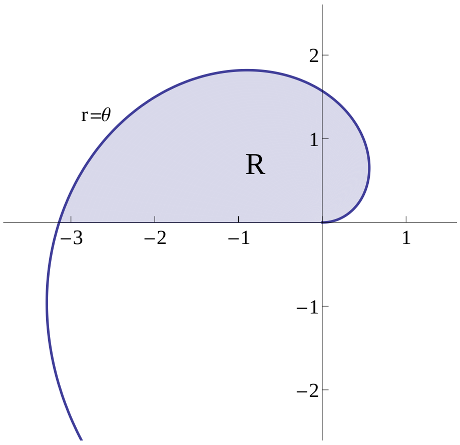
Soru 2 (Polar Coordinates)**.**

(a) [30p] Graph the curve

$$r = 1 + 2\cos\theta.$$



Let R be the region enclosed by the spiral $r = \theta$ for $0 \le \theta \le \pi$, as shown below.



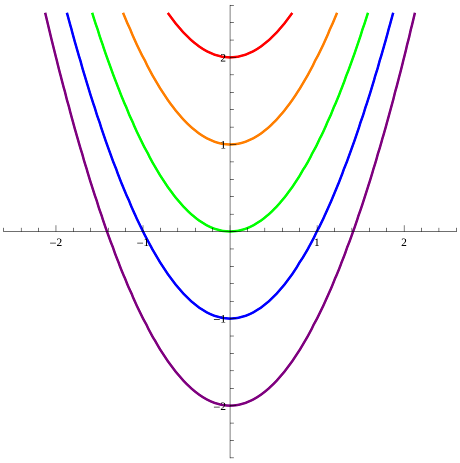
(b) [20p] Calculate the area of R.

The area of R is

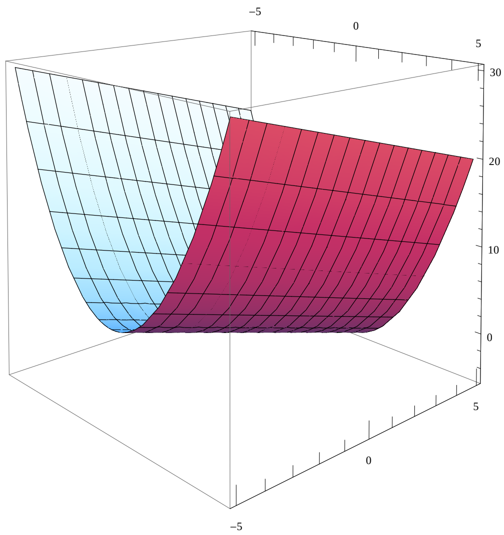
$$A = \int_0^\pi \frac{1}{2} r^2 \, d\theta = \int_0^\pi \frac{1}{2}\theta^2 \, d\theta = \left[\frac{1}{6}\theta^3\right]_0^\pi = \frac{\pi^3}{6}$$

Soru 3 (Functions of Several Variables)**.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 - y$.

(a) [10p] Plot the level curves $f(x, y) = -2$, $f(x, y) = -1$, $f(x, y) = 0$, $f(x, y) = 1$ and $f(x, y) = 2$ in $\mathbb{R}^2$. Label each level curve with the value of f.



(b) [16p] Sketch the surface $z = f(x, y)$ in $\mathbb{R}^3$.



Now let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(x, y, z) = e^{xz} \sin^2(x - 3y) + \cos z$.

(c) [8p] Calculate $\frac{\partial g}{\partial x}$.

$$g_x = ze^{xz} \sin^2(x - 3y) + 2e^{xz} \sin(x - 3y) \cos(x - 3y)$$

(d) [8p] Calculate $\frac{\partial g}{\partial y}$.

$$g_y = -6e^{xz} \sin(x - 3y) \cos(x - 3y)$$

(e) [8p] Calculate $\frac{\partial g}{\partial z}$.

$$g_z = xe^{xz} \sin^2(x - 3y) - \sin z.$$