Your Name

Your Signature

Student ID #

Professor's Name

Your Department

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

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1. $\boxed{\text{20 points}}$ Find all 2×2 diagonal matrices A that satisfy the equation $A^2-3A+2I=0$.

Solution: Let us choose $A=\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}$ where a and b are constants.

$$A^2-3A+2I=0$$
$$\begin{bmatrix} a^2 & 0\\ 0 & b^2\end{bmatrix}-\begin{bmatrix} 3a & 0\\ 0 & 3b\end{bmatrix}+\begin{bmatrix} 2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}$$
$$\begin{bmatrix} a^2-3a+2 & 0\\ 0 & b^2-3b+2\end{bmatrix}=\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}$$

When we solve the equations $a^2-3a+2=0$ and $b^2-3b+2=0$, we obtain $a=1,2$ and $b=1,2$. So, the following matrices satisfy the equation.

$$A_1=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},A_2=\begin{bmatrix} 2 & 0\\ 0 & 1\end{bmatrix},A_3=\begin{bmatrix} 1 & 0\\ 0 & 2\end{bmatrix},A_4=\begin{bmatrix} 2 & 0\\ 0 & 2\end{bmatrix}$$

2. $\boxed{\text{20 points}}$ Find the set of all solutions of the following system of linear equations.

$$\begin{aligned} v-2w+2x&=0\\ 2u-v+4w-3x&=0\\ 4u-v+6w-4x&=0\\ -2u+2v-6w+5x&=0 \end{aligned}$$

Solution: Let us write the augmented matrix of the system and reduce it to reduced row echelon form (RREF).

$$\left[\begin{array}{cccc|c} 0 & 1 & -2 & 2 & 0\\ 2 & -1 & 4 & -3 & 0\\ 4 & -1 & 6 & -4 & 0\\ -2 & 2 & -6 & 5 & 0 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 2 & -1 & 4 & -3 & 0\\ 0 & 1 & -2 & 2 & 0\\ 4 & -1 & 6 & -4 & 0\\ -2 & 2 & -6 & 5 & 0 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 2 & -1 & 4 & -3 & 0\\ 0 & 1 & -2 & 2 & 0\\ 0 & 1 & -2 & 2 & 0\\ 0 & 1 & -2 & 2 & 0 \end{array}\right] \backsim \left[\begin{array}{cccc|c} 2 & 0 & 2 & -1 & 0\\ 0 & 1 & -2 & 2 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has infinitely many solutions depending on two parameters.

$$\begin{aligned} 2u+2w-x&=0\\ v-2w+2x&=0 \end{aligned}$$

Let us choose w and x are free parameters. Therefore

$$\begin{aligned} x&=2t\\ w&=p\\ v&=2p-4t\\ u&=-p+t \end{aligned} \Rightarrow \begin{bmatrix} u\\ v\\ w\\ x \end{bmatrix} = \begin{bmatrix} -p+t\\ 2p-4t\\ p\\ 2t \end{bmatrix} = \begin{bmatrix} -1\\ 2\\ 1\\ 0 \end{bmatrix} p + \begin{bmatrix} 1\\ -4\\ 0\\ 2 \end{bmatrix} t$$

3. $\boxed{\text{20 points}}$ For which value(s) of k does the following system have

   a. no solutions?
   b. infinitely many solutions?
   c. one solution?

$$\begin{aligned} x+y+7z &= -7 \\ 2x+3y+17z &= 11 \\ x+2y+(k^2+1)z &= 6k \end{aligned}$$

Solution:

Let us reduce the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 2 & 3 & 17 & 11 \\ 1 & 2 & k^2+1 & 6k \end{array}\right] \backsim \left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & 25 \\ 0 & 1 & k^2-6 & 6k+7 \end{array}\right] \backsim$$

$$\left[\begin{array}{cccc} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & 25 \\ 0 & 0 & k^2-9 & 6k-18 \end{array}\right] \backsim \left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & 25 \\ 0 & 0 & (k-3)(k+3) & 6(k-3) \end{array}\right]$$

a. For $k=-3$, we obtain the system $\left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & 25 \\ 0 & 0 & 0 & -36 \end{array}\right]$. Therefore the system is inconsistent.

b. For $k=3$, we obtain the system $\left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & 25 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Therefore the system has infinitely many solutions depending on one parameter.

c. For $k \neq 3, (-3)$, we obtain the system $\left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & 25 \\ 0 & 0 & 1 & \dfrac{6}{k+3} \end{array}\right]$. Therefore the system has exactly one solution.

4. $\boxed{\text{20 points}}$ a. Compute A^{-3} where $A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$.

b. Let $B = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{bmatrix}$. Find an elementary matrix E that satisfies the equation $C = EB$

Solution:

a. $A^{-3} = \left(A^3\right)^{-1} = \left(A^{-1}\right)^3$. Let us calculate A^{-1}.

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{3.1-(-2)(-1)} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-3} = \left(A^{-1}.A^{-1}\right)A^{-1} = \left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\right)\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix}$$

b. The matrix B and the matrix C are row equivalent.

$$B = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{bmatrix}_{R_3-2R_1} \backsim \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{bmatrix} = C$$

Therefore the suitable elementary matrix is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

5. $\boxed{\text{20 points}}$ Calculate the inverse of $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & -2 \\ -4 & 2 & -8 \end{bmatrix}$

Solution: Let us reduce the matrix $[\ A \mid I\ ]$.

$$\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & -2 & 0 & 1 & 0 \\ -4 & 2 & -8 & 0 & 0 & 1 \end{array}\right] \backsim \left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & -2 & 0 & 1 & 0 \\ -4 & 2 & -8 & 0 & 0 & 1 \end{array}\right] \backsim \left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -10 & 2 & 1 & 0 \\ 0 & -10 & 8 & -4 & 0 & 1 \end{array}\right] \backsim$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 1 & -1 & \frac{2}{10} & \frac{1}{10} & 0 \\ 0 & 0 & -2 & -2 & 1 & 1 \end{array}\right] \backsim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -\frac{4}{10} & \frac{3}{10} & 0 \\ 0 & 1 & -1 & \frac{2}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \backsim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-14}{10} & \frac{8}{10} & \frac{5}{10} \\ 0 & 1 & 0 & \frac{12}{10} & -\frac{4}{10} & -\frac{5}{10} \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Therefore the inverse matrix of A is

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -14 & 8 & 5 \\ 12 & -4 & -5 \\ 10 & -5 & -5 \end{bmatrix}$$