Your Name

Your Signature

Student ID #

Professor's Name

Your Department

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total: | 100 | |

Do not write in the table to the right.

1. $\boxed{\text{20 points}}$ Calculate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ 0 & -1 & 5 & 7 \end{bmatrix}.$$

(You may wish to use appropriate row or column operations to simplify your calculations.)

**Solution:**

$$|A| = \begin{vmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ 0 & -1 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ -28 & -64 & -16 & 0 \end{vmatrix} = 0+0+1.(-1)^7 \begin{vmatrix} 2 & 2 & 1 \\ -1 & 0 & 3 \\ -28 & -64 & -16 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 2 & 1 \\ -1 & 0 & 3 \\ 36 & 0 & 16 \end{vmatrix} = -2.(-1)^3 \begin{vmatrix} -1 & 3 \\ 36 & 16 \end{vmatrix} = 2.(-16-108) = -248$$

$\det(A) = \boxed{\phantom{0}}$

2. $\boxed{\text{20 points}}$ Consider the linear system

$$\begin{aligned} x+2y-3z &= -4 \\ 4x-y+2z &= 8 \\ 2x+2y-3z &= -3. \end{aligned}$$

**<u>Use Cramer's Rule</u>** to find $y$.

**Solution:** Let us write the system as $A\mathbf{x}=\mathbf{b}$ where $A=\begin{bmatrix} 1 & 2 & -3 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$, $\mathbf{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} -4 \\ 8 \\ -3 \end{bmatrix}$. If $\det A \neq 0$ we can use Cramer's Rule. Let us calculate the $\det A$.

$$\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -9 & 14 \\ 0 & -2 & 3 \end{vmatrix} = \begin{vmatrix} -9 & 14 \\ -2 & 3 \end{vmatrix} = -27+28 = 1 \neq 0$$

We can use Cramer's rule to find variable $y$ and $y=\dfrac{|A_2|}{|A|}$. Therefore

$$|A_2| = \begin{vmatrix} 1 & -4 & -3 \\ 4 & 8 & 2 \\ 2 & -3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -4 & -3 \\ 0 & 24 & 14 \\ 0 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 24 & 14 \\ 5 & 3 \end{vmatrix} = 72-70 = 2$$

$$y = \frac{|A_2|}{|A|} = \frac{2}{1} = 2$$

$y = \boxed{\phantom{0}}$

3. Let $W$ be the set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x+y+z=0$.

   (a) $\boxed{\text{7 points}}$ Determine whether $W$ is a subspace of $\mathbb{R}^3$ or not. Verify your answer.

   (b) $\boxed{\text{7 points}}$ If $W$ is a subspace of $\mathbb{R}^3$, what is the dimension of $W$?

   (c) $\boxed{\text{6 points}}$ Write a basis for $W$.

**Solution:**

(a) Let us take the vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that $u_1+u_2+u_3=0$ and $v_1+v_2+v_3=0$. These vectors are in $W$.

1)

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}$$

and $u_1+v_1+u_2+v_2+u_3+v_3=0$ because of our assumption. That is, $\mathbf{u}+\mathbf{v}$ is in $W$.

2) $k.\mathbf{u} = \begin{bmatrix} ku_1 \\ ku_2 \\ ku_3 \end{bmatrix}$ and $ku_1+ku_2+ku_3 = k(u_1+u_2+u_3) = 0$. Therefore $k\mathbf{u}$ is in $\mathbf{W}$

That is, $W$ is a subspace of $\mathbb{R}^3$.

(b) The system $x+y+z=0$ involves 3 unknowns and 1 equation, therefore the system has infinitely many solutions depend on 2 parameters. Therefore $\dim W = 2$.

(c) $W$ involves vectors such that $x+y+z=0$. Let us try to express the vectors in $W$ as the linear combination of the linearly independent vectors. Let us take the variables $y$ and $z$ are free.

$$\begin{aligned} x &= -y-z = -t-p \\ y &= t \\ z &= p. \end{aligned}$$

Therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t-p \\ t \\ p \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} p$$

The set $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for $W$.

4. Let

$$p_1(x) = 1 + x + 2x^2 \qquad \text{and} \qquad p_2(x) = 5 - x + 4x^2.$$

(a) $\boxed{\text{10 points}}$ Is $p(x) = 2x + 3$ in $\text{span}\{p_1, p_2\}$?

(b) $\boxed{\text{10 points}}$ Is $q(x) = 3x^2 + 3$ in $\text{span}\{p_1, p_2\}$?

**Solution:**

If $p(x)$ and $q(x)$ are in $\text{span}\{p_1, p_2\}$, then $p(x)$ can be written as the linear combination of the $p_1$ and $p_2$. Therefore $p(x) = k_1 p_1(x) + k_2 p_2(x)$ and $q(x) = c_1 p_1(x) + c_2 p_2(x)$ where $k_1$, $k_2$, $c_1$ and $c_2$ are scalars.

$$\begin{aligned}
p(x) &= k_1 p_1(x) + k_2 p_2(x)\\
2x + 3 &= k_1 + k_1 x + 2k_1 x^2 + 5k_2 - k_2 x + 4k_2 x^2\\
2x + 3 &= (2k_1 + 4k_2)x^2 + (k_1 - k_2)x + (k_1 + 5k_2)
\end{aligned}$$

and

$$\begin{aligned}
q(x) &= c_1 p_1(x) + c_2 p_2(x)\\
3x^2 + 3 &= c_1 + c_1 x + 2c_1 x^2 + 5c_2 - c_2 x + 4c_2 x^2\\
3x^2 + 3 &= (2c_1 + 4c_2)x^2 + (c_1 - c_2)x + (c_1 + 5c_2)
\end{aligned}$$

The augmented matrix of the systems is as follows

$$\left[\begin{array}{cc|c|c} 1 & 5 & 3 & 3\\ 1 & -1 & 2 & 0\\ 2 & 4 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 5 & 3 & 3\\ 0 & -6 & -1 & -3\\ 0 & -6 & -6 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 5 & 3 & 3\\ 0 & -6 & -1 & -3\\ 0 & 0 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 0 & \frac{13}{6} & \frac{1}{2}\\ 0 & 1 & \frac{1}{6} & \frac{1}{2}\\ 0 & 0 & -5 & 0 \end{array}\right]$$

(a) According to the third row of the augmented matrix, we obtain $0k_1 + 0k_2 = -5$. Therefore the system has no solution. That is, $p(x)$ is not in $\text{span}\{p_1, p_2\}$.

(b) According to the reduced form of the augmented matrix, we obtain $c_1 = c_2 = \frac{1}{2}$. Therefore, $q(x)$ can be expressed as the linear combination of the polnomials $p_1(x)$ and $p_2(x)$

$$q(x) = \frac{1}{2} p_1(x) + \frac{1}{2} p_2(x)$$

That is, $q(x)$ is in $\text{span}\{p_1, p_2\}$.

5. $\boxed{\text{20 points}}$ Determine whether the set of matrices

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

forms a basis for the space of the $2 \times 2$ matrices $M_{22}$.

**Solution:** To decide whether the set $B = \{A_1, A_2, A_3, A_4\}$ forms a basis for $M_{22}$ we must answer two question: Is B a linearly independent set? Does B span $M_{22}$? For the first one we must solve the system

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 = 0$$

and for the second question we solve

$$A = c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let us solve the systems together. Thwe augmented matrix is

$$\left[\begin{array}{cccc|c|c} 1 & 0 & 1 & 0 & 0 & a \\ -1 & 1 & 0 & 1 & 0 & b \\ 0 & 1 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & d \end{array}\right]$$

Therefore $k_1 = k_2 = k_3 = k_4 = 0$, the set $B$ is linearly independent. And

$$\begin{aligned} c_1 &= c + d - b \\ c_2 &= c \\ c_3 &= a + b - c - d \\ c_4 &= d \end{aligned}$$

$M_{22}$ can be spanned by the matrices in B.