Your Name

Your Signature

Student ID #

Professor's Name

Your Department

- This exam is closed book.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. **Show your work in evaluating any limits, derivatives**.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Do not ask the invigilator anything.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. $\boxed{\text{20 points}}$ Let $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find a matrix P such that $D = P^{-1}AP$ where D is a diagonal matrix.

Solution:

First we calculate the eigenvalues and the corresponding eigenvectors of A.

$$0 = |A - \lambda I| = \begin{vmatrix} -2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = \lambda(-2-\lambda)(\lambda-2)$$

Therefore the eigenvalues of A are $\lambda_1 = -2, \lambda_2 = 0$, and $\lambda_3 = 2$. We calculate the corresponding eigenvectors by solving the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

$$\lambda_1 = -2 \implies (A+2I)\mathbf{u} = \mathbf{0} \implies \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \implies \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0 \implies A\mathbf{v} = \mathbf{0} \implies \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \implies \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_3 = 2 \implies (A-2I)\mathbf{w} = \mathbf{0} \implies \left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \implies \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

There are six correct answers to this question. One answer is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 1 & 3 & 4 & 4 \\ 2 & 4 & 6 & 0 \\ -1 & -3 & -4 & 5 \end{bmatrix}$.

   (a) $\boxed{\text{7 points}}$ Find a basis for the column space of A.

   (b) $\boxed{\text{7 points}}$ Find a basis for the null space of A.

   (c) $\boxed{\text{6 points}}$ Find the rank and nullity of A.

Solution: First we row reduce the matrix:

$$\begin{bmatrix} 1 & 3 & 4 & 4 \\ 2 & 4 & 6 & 0 \\ -1 & -3 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 4 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) The first, second and fourth columns of the reduced form of the matrix contain leading ones. So the first, second and fourth columns of the original matrix, A, form a basis for column space of A. Therefore

$$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \right\}.$$

(b) The null space of A is the solution set of the equation $A\mathbf{x} = \mathbf{0}$. The reduced form of the augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 4 & 0 \\ 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Therefore the system has infinitely many solutions depending on $4 - 3 = 1$ parameter. The general solution is

$$\mathbf{x} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vector set $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis for the null space of A.

(c) Since the rank of the matrix is equal to the dimension of the column space, we have $\text{rank}(A) = 3$
The nullity of A is the dimension of the null space of A. Thus $\text{nullity}(A) = 1$.

3. A box containing 10₺, 50₺, and 100₺ has 13 banknotes with a total value 830₺. The sum of the numbers of 10₺ valued banknotes and 50₺ valued banknotes is one greater than the number of the 100₺ valued banknotes.

   (a) $\boxed{\text{12 points}}$ Write a linear system with 3 equations in 3 unknowns for the number of banknotes in the box. [You must state what each of your variables represents.]

   (b) $\boxed{\text{8 points}}$ How many banknotes of each type are in the box?

Solution:

(a) Let us take the variables as follows

$$\begin{aligned} x &= \text{the number of 10₺ banknotes,} \\ y &= \text{the number of 50₺ banknotes, and} \\ z &= \text{the number of 100₺ banknotes.} \end{aligned}$$

Therefore, the required system of equations is

$$\begin{aligned} x+y+z &= 13 \\ 10x+50y+100z &= 830 \\ y+x &= z+1 \end{aligned}$$

(b) The augmented matrix for the system above is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 10 & 50 & 100 & 830 \\ 1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 0 & 40 & 90 & 700 \\ 0 & 0 & -2 & -12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 13 \\ 0 & 4 & 9 & 70 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

When we use backward substitution, we obtain $z = 6, y = 4$ and $x = 3$.

4. (a) $\boxed{\text{10 points}}$ Calculate $\operatorname{adj}(A)$ where $A^{-1} = \begin{bmatrix} 10 & -1 & 3 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

   (b) $\boxed{\text{10 points}}$ Let A, B, C and D be 3×3 matrices where $\det A = 2, \det B = 5, \det C = 8$ and $\det D = 12$. Find $\det(5A^4B^{-3}C^{-2T}D^T)$.

Solution:

(a) Remember that $A^{-1} = \dfrac{1}{\det A}\operatorname{adj}(A)$ and $\det A = \dfrac{1}{\det A^{-1}}$. Therefore $\operatorname{adj}(A) = \det A A^{-1} = \dfrac{1}{\det A^{-1}} A^{-1}$.

Since

$$\det A^{-1} = \begin{vmatrix} 10 & -1 & 3 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 2 & 0 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 0 + 1(-1)^{2+3} \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} + 0 = -6$$

we have that

$$\operatorname{adj}(A) = \det A A^{-1} = \frac{1}{\det A^{-1}} A^{-1} = -\frac{1}{6} \begin{bmatrix} 10 & -1 & 3 \\ 4 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

(b)

$$\begin{aligned} \det(5A^4B^{-3}C^{-2T}D^T) &= 5^3(\det A)^4(\det B^{-1})^3(\det C^{-T})^2 \det D^T \\ &= 5^3(\det A)^4 \frac{1}{(\det B)^3} \frac{1}{(\det C)^2} \det D = 5^3 2^4 \frac{1}{5^3} \frac{1}{8^2} 12 = 3 \end{aligned}$$

5. Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ where

$$\mathbf{u}_1 = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

(a) $\boxed{\text{8 points}}$ Find the transition matrix from B to S.

(b) $\boxed{\text{6 points}}$ Compute the coordinate vector $[\mathbf{w}]_B$ where $\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$.

(c) $\boxed{\text{6 points}}$ Compute the coordinate vector $[\mathbf{w}]_S$ by using transition matrix $P_{B\to S}$.

Solution:

(a) The matrix $\left[S|B\right]$ is row equivalent to the matrix $[I|P_{B\to S}]$. Therefore

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -3 & -3 & 1 \\ 1 & -1 & -3 & 0 & 2 & 6 \\ 0 & 3 & 1 & -3 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -3 & -3 & 1 \\ 0 & 1 & -1 & 3 & 5 & 5 \\ 0 & 3 & 1 & -3 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 3 & 7 & 11 \\ 0 & 1 & -1 & 3 & 5 & 5 \\ 0 & 0 & 4 & -12 & -16 & -16 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 3 & 7 & 11 \\ 0 & 1 & -1 & 3 & 5 & 5 \\ 0 & 0 & 1 & -3 & -4 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9 & -9 & -3 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3 & -4 & -4 \end{array}\right]$$

The transition matrix is $P_{B\to S} = \begin{bmatrix} -9 & -9 & -3 \\ 0 & 1 & 1 \\ -3 & -4 & -4 \end{bmatrix}$

(b) Let us find the coordinates of $\mathbf{w}$ relative to the basis B. We solve the following system

$$\left[\begin{array}{ccc|c} -3 & -3 & 1 & -5 \\ 0 & 2 & 6 & 8 \\ -3 & -1 & -1 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & -3 & 1 & -5 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -3 & -3 & 1 & -5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -8 & -8 \end{array}\right]$$

Therefore $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) We can find $[\mathbf{w}]_S] = P_{B\to S}[\mathbf{w}]_B]$. Let us calculate

$$[\mathbf{w}]_S] = P_{B\to S}[\mathbf{w}]_B]$$

$$\Rightarrow [\mathbf{w}]_S = \begin{bmatrix} -9 & -9 & -3 \\ 0 & 1 & 1 \\ -3 & -4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 2 \\ -11 \end{bmatrix}$$