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- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Problem	Points	Score
1	20	
2	15	
3	20	
4	25	
5	20	
Total:	100	

Do not write in the table to the right.

1. (a) $\boxed{\text{10 points}}$ Show that if a square matrix A satisfies the equation $A^2+5A-2I=0$, then $A^{-1}=\frac{1}{2}(A+5I)$.

Solution: If there exists the B matrix such that $AB=BA=I$, then B is called the inverse of A. Additionally, the matrix A satisfies the equation $A^2+5A-2I=0$, so $I=\frac{1}{2}(A^2+5A)$.

$$\frac{1}{2}(A+5I).A=\frac{1}{2}(A^2+5A)=I$$
$$A.\frac{1}{2}(A+5I)=\frac{1}{2}(A^2+5A)=I$$

Therefore, $A^{-1}=\frac{1}{2}(A+5I)$ is the inverse matrix of A.

(b) $\boxed{\text{10 points}}$ Find an elementary matrix E that satisfies equation $EA=B$ where $A=\begin{bmatrix}2&6&-8\\0&5&3\\4&7&9\end{bmatrix}$ and $B=\begin{bmatrix}2&6&-8\\0&5&3\\0&-5&25\end{bmatrix}$.

Solution: The matrices A and B are row equivalent matrices.

$$A = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 4 & 7 & 9 \end{bmatrix} \underset{R_3 - 2R_1}{\rightarrow} \begin{bmatrix} 2 & 6 & -8 \\ 0 & 5 & 3 \\ 0 & -5 & 25 \end{bmatrix} = B$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow EA = B$$

2. $\boxed{\text{15 points}}$ Suppose that $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\det A = (-4)$. Use Cramer's rule to find x_3.

Solution: The determinant of the matrix A is different from zero, so we can use Cramer's rule to solve the system. We can find x_3 by calculating $x_3 = \dfrac{|A_3|}{|A|}$.

$$|A_3| = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \end{vmatrix} = 2(-1)^{3+1} \begin{vmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 2 \begin{vmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2.2(-1)^{3+3} \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -8$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{-4} = 2$$

3. Let $A = \begin{bmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 3 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 2 & 0 & 2 & 0 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. A and D are similiar matrices.

(a) $\boxed{\text{6 points}}$ Find the determinant of A.

(b) $\boxed{\text{8 points}}$ Find the eigenvalues of A.

(c) $\boxed{\text{6 points}}$ Find the nullity and the rank of A.

Solution: A and D are similiar matrices. Therefore, their determinant, eigenvalues, nullity and rank are same.

(a) $|A| = |D| = 6.5.0.0.0 = 0$

(b) The eigenvalues of D are $\lambda_1 = 6, \lambda_2 = 5$ and $\lambda_3 = \lambda_4 = \lambda_5 = 0$,so the eigenvalues of A are $\lambda_1 = 6, \lambda_2 = 5$ and $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

(c) The reduced row echelon form of the D is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced matrix has two non zero rows, so the rank and the nullity of D and A are 2 and 3, respectively.

4. Let $T : P_3 \to P_2$ be the transformation defined by $T(p(x)) = 2p'(x) - p''(x)$.
   (a) $\boxed{\text{5 points}}$ Show that T is a linear transformation.
   (b) $\boxed{\text{5 points}}$ Find the standard matrix of the transformation relative to the standard basis.
   (c) $\boxed{\text{8 points}}$ Find a basis for $\ker T$ $(i.e., NullA)$.
   (d) $\boxed{\text{7 points}}$ Find a basis for range (image) of T.

Solution: Suppose that $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ in P_3. The coordinates of $p(x)$ and its image relative to the standard basis are written as

$$T(p(x)) = 2(a_1 + 2a_2x + 3a_3x^2) - (2a_2 + 6a_3x) = (2a_1 - 2a_2) + (4a_2 + 6a_3)x + 6a_3x^2$$

$$T\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 2a_1 - 2a_2 \\ 4a_2 + 6a_3 \\ 6a_3 \end{bmatrix}$$

(a) Suppose that $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ are in P_3 and $k \in \mathbb{R}$.

$$T(p(x)+q(x)) = T\left(\begin{bmatrix} a_0+b_0 \\ a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}\right) = \begin{bmatrix} 2a_1+2b_1-2a_2-2b_2 \\ 4a_2+4b_2+6a_3+6b_3 \\ 6a_3+6b_3 \end{bmatrix} = \begin{bmatrix} 2a_1-2a_2 \\ 4a_2+6a_3 \\ 6a_3 \end{bmatrix} + \begin{bmatrix} 2b_1-2b_2 \\ 4b_2+6b_3 \\ 6b_3 \end{bmatrix} = T(p(x)) + T(q(x))$$

$$T(kp(x)) = T\left(\begin{bmatrix} ka_0 \\ ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}\right) = \begin{bmatrix} 2ka_1-2ka_2 \\ 4ka_2+6ka_3 \\ 6ka_3 \end{bmatrix} = k\begin{bmatrix} 2a_1-2a_2 \\ 4a_2+6a_3 \\ 6a_3 \end{bmatrix} = kT(p(x))$$

Therefore it is a linear transformation.

(b) $$T\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 2a_1-2a_2 \\ 4a_2+6a_3 \\ 6a_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

(c) $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Therefore, $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ is a basis for the solution space of the $A\mathbf{x} = \mathbf{0}$ and $p_1(x) = 1$ is a basis for $\ker T$.

(d) The set $\left\{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix}\right\} \Rightarrow \begin{array}{rcl} q_1(x) & = & 2 \\ q_2(x) & = & -2+4x \\ q_3(x) & = & 6x+6x^2 \end{array}$ is a basis for the image of T relative to the stadart basis.

5. $\boxed{\text{20 points}}$ The eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}$ corresponding to $\lambda_1 = -2$ and $\lambda_2 = \lambda_3 = 4$ are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, respectively. Find a matrix Q that **orthogonally** diagonalizes A such that $Q^T Q = I$

Solution: The columns of Q are the orthonormal eigenvectors of A. We can find an orthogonal basis for eigenspace of A.

$$u_1 = v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = v_2 - \frac{< v_2, u_1 >}{||u_1||^2} u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{3} u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$u_3 = v_3 - \frac{< v_3, u_1 >}{||u_1||^2} u_1 - \frac{< v_3, u_2 >}{||u_2||^2} u_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

u_1, u_2 and u_3 are orthogonal vectors.

$$q_1 = \frac{u_1}{||u_1||} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{u_2}{||u_2||} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$q_3 = \frac{u_3}{||u_3||} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

are orthonormal. $Q = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$