Your Name / İsim Soyisim

Your Signature / İmza

Student ID # / Öğrenci Numarası

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Professor’s Name / Öğretim Üyesi

Your Department / Bölüm

- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total: | 100 | |

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1. $\boxed{\text{20 points}}$ Find the set of all solutions of the system $\begin{array}{rl} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{array}$ by using augmented matrix.

**Solution:**

The augmented matrix of the system is $\left[\begin{array}{ccccc|c} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{array}\right]$. Let us find the reduced row echelon form of the augmented matrix.

$$\left[\begin{array}{ccccc|c} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{array}\right] \sim \left[\begin{array}{ccccc|c} -3 & -1 & 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & -3 & 1 & 0 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{7}{6} & -\frac{7}{6} & \frac{5}{3} \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & -3 & 4 \end{array}\right]$$

The system has infinitely many solutions depend on two parameters. Let us take $x_4 = s$ and $x_5 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} - \frac{7}{6}s + \frac{7}{6}t \\ \frac{3}{2}s - \frac{1}{2}t \\ 4 - 2s + 3t \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -\frac{7}{6} \\ \frac{3}{2} \\ -2 \end{bmatrix} s + \begin{bmatrix} \frac{7}{6} \\ -\frac{1}{2} \\ 3 \end{bmatrix} t.$$

2. $\boxed{\text{20 points}}$ Find for what values of $k$ the system $\begin{array}{l} x+y+kz=1 \\ x+ky+z=1 \\ kx+y+z=1 \end{array}$ has

   (a) no solution.

   (b) infinitely many solutions.

   (c) a unique solution.

   **Use augmented matrix of the system.**

**Solution:** Let us reduce the augmented matrix of the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & 1-k \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & (2+k)(1-k) & 1-k \end{array}\right]$$

(a) If $k=-2$, then reduced form of the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$. Therefore, if $k=-2$, the system has no solution.

(b) If $k=1$, then we obtain $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. The system has infinitely many solutions depend on two parameters.

(c) If $k \neq -2$ and $k \neq 1$,then $\left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{k+2} \end{array}\right]$. The system has exactly one solution.

3. $\boxed{\text{20 points}}$ Suppose that $(I+2A)^{-1} = \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}$. Find A.

**Solution:** Remember that $(A^{-1})^{-1} = A$. Therefore

$$(I+2A)^{-1} = \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}$$

$$I+2A = \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{(-1).5-3.4} \begin{bmatrix} 5 & -3 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{17} & \frac{3}{17} \\ \frac{4}{17} & \frac{1}{17} \end{bmatrix}$$

$$2A = \begin{bmatrix} -\frac{5}{17} & \frac{3}{17} \\ \frac{4}{17} & \frac{1}{17} \end{bmatrix} - I = \begin{bmatrix} -\frac{5}{17} & \frac{3}{17} \\ \frac{4}{17} & \frac{1}{17} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{22}{17} & \frac{3}{17} \\ \frac{4}{17} & -\frac{16}{17} \end{bmatrix}$$

$$A = \frac{1}{2} \begin{bmatrix} -\frac{22}{17} & \frac{3}{17} \\ \frac{4}{17} & -\frac{16}{17} \end{bmatrix} = \begin{bmatrix} -\frac{11}{17} & \frac{3}{34} \\ \frac{2}{17} & -\frac{8}{17} \end{bmatrix}$$

4. $\boxed{\text{20 points}}$ Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$. Calculate the $\begin{vmatrix} 3g & 3h & 3i \\ 2a+d & 2b+e & 2c+f \\ d & e & f \end{vmatrix}$.

**Solution:**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-6) \Rightarrow \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = 6 \Rightarrow \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = (-6)$$

$$\begin{vmatrix} 3g & 3h & 3i \\ a & b & c \\ d & e & f \end{vmatrix} = (-6).3 \Rightarrow \begin{vmatrix} 3g & 3h & 3i \\ 2a & 2b & 2c \\ d & e & f \end{vmatrix} = (-18).2$$

$$\begin{vmatrix} 3g & 3h & 3i \\ 2a+d & 2b+e & 2c+f \\ d & e & f \end{vmatrix} = (-36)$$

5. $\boxed{\text{20 points}}$ Evaluate the determinant of the matrix $A = \begin{bmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ 0 & -1 & 5 & 7 \end{bmatrix}$

**Solution:**

$$|A| = \begin{vmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ 0 & -1 & 5 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 9 & 3 & 1 \\ -28 & -64 & -16 & 0 \end{vmatrix}$$

$$= 1.(-1)^{(3+4)} \begin{vmatrix} 2 & 2 & 1 \\ -1 & 0 & 3 \\ -28 & -64 & -16 \end{vmatrix} = (-1) \begin{vmatrix} 2 & 2 & 7 \\ -1 & 0 & 0 \\ -28 & -64 & -100 \end{vmatrix}$$

$$= (-1)(-1)(-1)^{2+1} \begin{vmatrix} 2 & 7 \\ -64 & -100 \end{vmatrix}$$

$$= (-1)[2(-100) - 7(-64)]$$

$$= -248$$