Your Name / İsim Soyisim

Your Signature / İmza

Student ID # / Öğrenci Numarası

Professor's Name / Öğretim Üyesi

Your Department / Bölüm

- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (a) $\boxed{\text{10 points}}$ Let V be the set of vectors of the form $\begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. Is V a subspace of $\mathbb{R}^3$? Why?

Solution: Suppose that $\mathbf{u} = \begin{bmatrix} u \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v \\ 1 \\ 0 \end{bmatrix}$ are two vectors in V, and k is a scalar

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} v \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} u+v \\ 2 \\ 0 \end{bmatrix} \notin V \Rightarrow V \text{is not a subspace of} \mathbb{R}^3$$

(b) $\boxed{\text{10 points}}$ Let W be the set of vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $c = a - b$ in $\mathbb{R}^3$. Is W a subspace of $\mathbb{R}^3$? Why?

Solution: Suppose that the vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_1 - u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_1 - v_2 \end{bmatrix}$ are in W, and k is a scalar.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_1 - u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_1 - v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_1 - u_2 + v_1 - v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ (u_1 + v_1) - (u_2 + v_2) \end{bmatrix} \in W$$

$$k\mathbf{u} = \begin{bmatrix} ku_1 \\ ku_2 \\ k(u_1 - u_2) \end{bmatrix} \in W \Rightarrow W \text{is a subspace of} \mathbb{R}^3.$$

2. $\boxed{\text{20 points}}$ Suppose that $\mathbf{p}_1(x) = 2 - x + 4x^2, \mathbf{p}_2(x) = 3 + 6x + 2x^2, \mathbf{p}_3(x) = -15x + 8x^2$. Is the set $S = \{\mathbf{p}_1, \mathbf{p}_2, mathbf p_3(x)\}$ a linearly dependent set? If the answer is **yes**, then find the dependency equation.

Solution: We must find the solution set of the $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = 0$ to decide the behaviour of the given polynomials.

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ -1 & 6 & -15 & 0 \\ 4 & 2 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 15 & -30 & 0 \\ -1 & 6 & -15 & 0 \\ 0 & 26 & -52 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -6 & 15 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 26 & -52 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -6 & 15 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_2 - 2c_3 = 0 \text{and} c_1 - 6c_2 + 15c_3 = 0 \Rightarrow \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} t$$

The polynomials are linearly dependent and the dependency equation is $-3\mathbf{p}_1 + 2\mathbf{p}_2 + \mathbf{p}_3 = 0$ by taking $t = 1$.

3. Suppose that $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$.

   (a) $\boxed{\text{14 points}}$ Find a basis for the null space of A.

   (b) $\boxed{\text{6 points}}$ Find the nullity and rank of A.

Solution:

(a) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow x_2 + x_3 + \frac{4}{7}x_4 = 0 \Rightarrow x_2 = -p - 4k, x_3 = p, x_4 = 7k$

$\Rightarrow x_1 + 4x_2 + 5x_3 + 2x_4 = 0 \Rightarrow x_1 = -4(-p-4k) - 5p - 2(7k) = -p + 2k$

Therefore $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -p+2k \\ -p-4k \\ p \\ 7k \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} p + \begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix} k$

The set $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix} \right\}$ is a basis for the solution space of $A\mathbf{x} = \mathbf{0}$, so it is a basis for the null space of A.

(b) The nullity of A is the dimension of the null space of A. Therefore, $Null(A) = 2$.

$$Null(A) + Rank(A) = \text{The column number of} A$$

then $Rank(A) = 4 - 2 = 2$.

4. Let $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ be two bases for $\mathbb{R}^2$ where $\mathbf{u}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\3\end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\4\end{bmatrix}$.

   (a) 7 points Find the coordinates of $\mathbf{w} = \begin{bmatrix}3\\-3\end{bmatrix}$ relative to the basis B.

   (b) 8 points Find the transition matrix $P_{B\to S}$ from B to S.

   (c) 5 points Find the coordinates of $\mathbf{w}$ relative to the basis S by using the transition matrix.

Solution: Let us take $B = \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}, S = \begin{bmatrix}1 & 1\\3 & 4\end{bmatrix}$.

(a) The coordinates of $\mathbf{w} = \begin{bmatrix}3\\-3\end{bmatrix}$ relative to the basis B, $[\mathbf{w}]_B$, is the solution set of $B\,[\mathbf{w}]_B = \mathbf{w}$.

$$[\mathbf{w}]_B = B^{-1}\mathbf{w} = \frac{1}{3-4}\begin{bmatrix}3 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}3\\-3\end{bmatrix} = \begin{bmatrix}-15\\9\end{bmatrix}$$

(b) The transition matris from B to S can be calculated by using $P_{B\to S} = S^{-1}B$.

$$S^{-1} = \frac{1}{4-3}\begin{bmatrix}4 & -1\\-3 & 1\end{bmatrix} \Rightarrow P_{B\to S} = S^{-1}B = \begin{bmatrix}4 & -1\\-3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix} = \begin{bmatrix}2 & 5\\-1 & -3\end{bmatrix}$$

or we can find the transition matrix by using $[S|B] \sim [\,I\,|P_{B\to S}]$.

$$[S|B] = \left[\begin{array}{cc|cc}1 & 1 & 1 & 2\\3 & 4 & 2 & 3\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 1 & 1 & 2\\0 & 1 & -1 & -3\end{array}\right] \sim \left[\begin{array}{cc|cc}1 & 0 & 2 & 5\\0 & 1 & -1 & -3\end{array}\right] = [\,I\,|P_{B\to S}]$$

(c) $[\mathbf{w}]_S = P_{B\to S}\,[\mathbf{w}]_B = \begin{bmatrix}2 & 5\\-1 & -3\end{bmatrix}\begin{bmatrix}-15\\9\end{bmatrix} = \begin{bmatrix}15\\-12\end{bmatrix}$

5. Suppose that $A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(a) $\boxed{\text{6 points}}$ Find the eigenvalues of A.

(b) $\boxed{\text{9 points}}$ Find the corresponding eigenvectors of A.

(c) $\boxed{\text{5 points}}$ Is A diagonalizable? Find a matrix P that diagonalize A, if possible.

Solution:

(a) $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & -1 & 0 \\ -2 & 4-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)\,[(3-\lambda)(4-\lambda)-2] = (1-\lambda)(5-\lambda)(2-\lambda) = 0$

The eigenvalues of A are $\lambda_1 = 5, \lambda_2 = 2, \lambda_3 = 1$.

(b) To find the corresponding eigenvectors of A, we solve the homogeneous system $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

$$\lambda_1 = 5 \Rightarrow (A-5I)\mathbf{u} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2 \Rightarrow (A-2I)\mathbf{v} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 1 \Rightarrow (A+2I)\mathbf{w} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

(c) $P = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$