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DEPARTMENT:

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SIGNATURE:

| Question | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 25 | |
| 3 | 25 | |
| 4 | 30 | |
| Total: | 100 | |

- The time limit is 70 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. Suppose that
   - $V$ is a vector space;
   - $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a basis for $V$;
   - $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $V$;
   - $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$;
   - $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$;
   - $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$; and
   - $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

   (a) $\boxed{\text{10 points}}$ Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

   **Solution:** Note first that $[\mathbf{a}_1]_\mathcal{B} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$, $[\mathbf{a}_2]_\mathcal{B} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $[\mathbf{a}_3]_\mathcal{B} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$. Hence

   $$P_{\mathcal{A}\to\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{A}}{P} = \big[[\mathbf{a}_1]_\mathcal{B} \quad [\mathbf{a}_2]_\mathcal{B} \quad [\mathbf{a}_3]_\mathcal{B}\big] = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

   (b) $\boxed{\text{10 points}}$ Find $[\mathbf{x}]_\mathcal{B}$

   **Solution:**

   $$[\mathbf{x}]_\mathcal{B} = P_{\mathcal{A}\to\mathcal{B}}[\mathbf{x}]_\mathcal{A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}.$$

2. Let $W = \left\{ \begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} \middle| \ a,b,c \in \mathbb{R} \right\}$.

(a) i. $\boxed{\text{5 points}}$ Is $W$ a subspace of $\mathbb{R}^4$? [You must justify (explain) your answer.]

ii. $\boxed{\text{10 points}}$ Find a set of vectors that spans $W$.

**Solution:** We can answer parts (i) and (ii) together. Since

$$\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} = a \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

we have that $W = \operatorname{span} \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

A span of a set of vectors in $V$ is always a subspace of $V$. Hence $W$ is a subspace of $\mathbb{R}^4$.

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(b) $\boxed{\text{10 points}}$ Now let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$ and $H = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find a basis for $H$.

**Solution:**

METHOD 1: Since $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$, we can see that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not linearly independent. We can remove $\mathbf{v}_3$, say, to be left with $\{\mathbf{v}_1, \mathbf{v}_2\}$. Clearly $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel. Hence $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $H$.

METHOD 2: Writing these three vectors as rows of a matrix gives

$$\begin{bmatrix} 4 & -3 & 7 \\ 1 & 9 & -2 \\ 7 & 11 & 6 \end{bmatrix}.$$

Row reducing this matrix gives

$$\begin{bmatrix} 1 & 0 & \frac{19}{13} \\ 0 & 1 & -\frac{5}{13} \\ 0 & 0 & 0 \end{bmatrix}.$$

Recall that the row space of a matrix is unchanged under row operations. Hence $\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{19}{13} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{5}{13} \end{bmatrix} \right\}$ is a basis for $H$.

3. Let $A = \begin{bmatrix} -2 & 5 & -1 \\ 6 & -10 & 8 \\ 10 & -18 & 12 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$.

[You must justify (explain) your answers. Col $A$ = "the column space of $A$". Nul $A$ = "the null space of $A$".]

(a) $\boxed{\text{13 points}}$ Is $\mathbf{w}$ in Col $A$? Why?

**Solution:** An equivalent question is: "Is $A\mathbf{x} = \mathbf{w}$ consistent?" If we write an augmented matrix, then row reduce it,

$$\begin{bmatrix} -2 & 5 & -1 & 3 \\ 6 & -10 & 8 & 1 \\ 10 & -18 & 12 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 5 & -1 & 3 \\ 0 & 5 & 5 & 10 \\ 10 & -18 & 12 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 5 & -1 & 3 \\ 0 & 5 & 5 & 10 \\ 0 & 7 & 7 & 14 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 5 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 7 & 7 & 14 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 5 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{5}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & \frac{7}{2} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can see that $A\mathbf{x} = \mathbf{w}$ is consistent. Hence $\mathbf{w} \in \operatorname{Col} A$.

(b) $\boxed{\text{12 points}}$ Is $\mathbf{w}$ in Nul $A$? Why?

**Solution:**
Since

$$A\mathbf{w} = \begin{bmatrix} -2 & 5 & -1 \\ 6 & -10 & 8 \\ 10 & -18 & 12 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6+5+1 \\ 18-10-8 \\ 30-18-12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

the answer is "yes, $\mathbf{w}$ is in Nul $A$".

4. Let $A = \begin{bmatrix} 4 & 8 & -5 & 23 & 3 \\ 3 & 6 & -10 & 26 & -1 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 5 & -3 & 90 \\ 0 & 0 & 1 & -\frac{7}{5} & -13 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. You may assume that $A$ is row equivalent to $B$.

(a) $\boxed{\text{12 points}}$ Find a basis for Col $A$.

**Solution:** Since $B$ has leading ones (pivot positions) in its first, third and fifth columns, and since $A$ and $B$ are row equivalent, the first, third and fifth columns of $A$ will give a basis for Col $A$. Therefore

$$\left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -10 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

is a basis for Col $A$.

(b) $\boxed{\text{3 points}}$ Find rank($A$).

**Solution:** $\text{rank}(A) = \dim(\text{Col}\,A) = (\text{number of vectors in a basis for Col}\,A) = 3.$

(c) $\boxed{\text{3 points}}$ Find nullity($A$) = dim(Nul $A$).

**Solution:** $\text{nullity}(A) = (\text{number of columns in } A) - \text{rank}(A) = 5 - 3 = 2.$

(d) $\boxed{\text{12 points}}$ Find a basis for Nul $A$.

**Solution:** By (c), our basis must contain 2 vectors. Recall that the null space of a matrix is unchanged under row operations. We can row reduce $B$ to

$$C = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} = C\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 + 4x_4 \\ x_3 - \frac{7}{5}x_4 \\ x_5 \\ 0 \end{bmatrix}$$

gives

$$\mathbf{x} = \begin{bmatrix} -2s - 4t \\ s \\ \frac{7t}{5} \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}.$$

Hence $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for Nul $A$.