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SIGNATURE:

Question	Points	Score
1	20	
2	25	
3	25	
4	30	
Total:	100	

- The time limit is 90 minutes.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

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1. $\boxed{\text{20 points}}$ Consider the linear system

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 3 \\ -3x_1 \qquad + 2x_3 \qquad &= 0 \\ x_2 - 2x_3 \qquad &= 2 \\ -3x_1 + x_2 \qquad\quad + x_4 &= 3. \end{aligned}$$

<u>**Use Cramer's Rule**</u> to find x_1.

Solution: Since

$$\det(A) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ -3 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} -3 & 2 \\ 0 & -2 \end{vmatrix} = -2 - 6 = -8$$

and

$$\det(A_1) = \begin{vmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{vmatrix} = -2\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -2,$$

we have that

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-2}{-8} = \frac{1}{4}$$

by Cramer's Rule.

You can check that $\mathbf{x} = \left(\frac{1}{4}, \frac{11}{4}, \frac{3}{8}, 1\right)$ if you wish.

2. Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. The eigenvalues of A are $\lambda = 2$ and $\lambda = 8$.

(a) 7 points Is A diagonalisable? Why?

Solution: Since A is 3×3 but only has two distinct eigenvalues, we must calculate the eigenvectors of A. First let $\lambda = 2$. Since

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1-2x_2-2x_3 \\ -2x_1-2x_2-2x_3 \\ -2x_1-2x_2-2x_3 \end{bmatrix},$$

we have two free variables. Let $x_2 = t$ and $x_3 = s$. Then $x_1 = -x_2 - x_3 = -t - s$. Hence

$$\mathbf{x} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = s\mathbf{p}_1 + t\mathbf{p}_2.$$

Now consider $\lambda = 8$. Since

$$\mathbf{0} = (\lambda I - A)\mathbf{x} = \begin{bmatrix} \lambda-4 & -2 & -2 \\ -2 & \lambda-4 & -2 \\ -2 & -2 & \lambda-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1-2x_2-2x_3 \\ -2x_1+4x_2-2x_3 \\ -2x_1-2x_2+4x_3 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we have only one free variable. Let $x_3 = r$. Then $x_2 = x_3 = r$ and $x_1 = \frac{1}{2}(x_2 + x_3) = r$ also. Hence

$$\mathbf{x} = \begin{bmatrix} r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r\mathbf{p}_3.$$

Finally, A is diagonalisable since $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent.

(b) 18 points If A is diagonalisable, then

i. find a matrix P which diagonalises A;
ii. find P^{-1}; and
iii. calculate $P^{-1}AP$.

Solution:

i. $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

ii. Since

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

we have that $P^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

iii. $P^{-1}AP = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -2 & -2 & 8 \\ 0 & 2 & 8 \\ 2 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$.

3. Let $B = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$. The characteristic polynomial of B is $-\lambda^3 + 14\lambda^2 - 57\lambda + 72$. Note that $\lambda = 8$ is an eigenvalue of B.

(a) $\boxed{\text{5 points}}$ Find all of the eigenvalues of B.

Solution: Since we are told that $\lambda = 8$ is an eigenvalue of B, we must have that $-\lambda^3 + 14\lambda^2 - 57\lambda + 72 = (\lambda - 8)(a\lambda^2 + b\lambda + c)$ for some $a, b, c \in \mathbb{R}$. We calculate that

$$\begin{aligned} 0 &= -\lambda^3 + 14\lambda^2 - 57\lambda + 72 \\ &= (\lambda - 8)(a\lambda^2 + b\lambda + c) \\ &= a\lambda^3 + b\lambda^2 + c\lambda - 8a\lambda^2 - 8b\lambda - 8c \\ &= a\lambda^3 + (b - 8a)\lambda^2 + (c - 8b)\lambda - 8c \qquad (a = -1,\ b = 6,\ c = -9) \\ &= (\lambda - 8)(-\lambda^2 + 6\lambda - 9) \\ &= -(\lambda - 8)(\lambda^2 - 6\lambda + 9) \\ &= -(\lambda - 8)(\lambda - 3)(\lambda - 3). \end{aligned}$$

It is then clear that the eigenvalues of B are 8 and 3.

(b) $\boxed{\text{20 points}}$ Find bases for the eigenspaces of B.

Solution: First consider $\lambda = 8$. Then

$$\mathbf{0} = (\lambda I - B)\mathbf{x} = \begin{bmatrix} \lambda - 4 & -2 & -3 \\ 1 & \lambda - 1 & 3 \\ -2 & -4 & \lambda - 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -3 \\ 1 & 7 & 3 \\ -2 & -4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 - 2x_2 - 3x_3 \\ x_1 + 7x_2 + 3x_3 \\ -2x_1 - 4x_2 - x_3 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 4 & -2 & -3 & 0 \\ 1 & 7 & 3 & 0 \\ -2 & -4 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 & 0 \\ 4 & -2 & -3 & 0 \\ -2 & -4 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 & 0 \\ 0 & -30 & -15 & 0 \\ 0 & 10 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we have one free variable. Say $x_3 = 2r$. Then $x_2 = -\frac{1}{2}x_3 = -r$ and $x_1 = -7x_2 - 3x_3 = 7r - 6r = r$. Therefore

$$\mathbf{x} = \begin{bmatrix} r \\ -r \\ 2r \end{bmatrix} = r \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Now consider $\lambda = 3$. Then

$$\mathbf{0} = (\lambda I - B)\mathbf{x} = \begin{bmatrix} \lambda - 4 & -2 & -3 \\ 1 & \lambda - 1 & 3 \\ -2 & -4 & \lambda - 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ -2 & -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 - 2x_2 - 3x_3 \\ x_1 + 2x_2 + 3x_3 \\ -2x_1 - 4x_2 - 6x_3 \end{bmatrix}.$$

Since

$$\begin{bmatrix} -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \\ -2 & -4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we have two free variables. Let $x_2 = s$ and $x_3 = t$. Then $x_1 = -2x_2 - 3x_3 = -2s - 3t$. Therefore

$$\mathbf{x} = \begin{bmatrix} -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ are bases for the eigenspaces corresponding to $\lambda = 8$ and $\lambda = 3$ respectively.

4. (a) $\boxed{\text{5 points}}$ Consider the function $S : \mathbb{R}^2 \to \mathbb{R}^3$, $S(x, y) = (x + y, x - y, 1)$. Is S a linear transformation? You must justify your answer.

Solution: No. All linear transformations map $\mathbf{0}$ to $\mathbf{0}$. Since $S(\mathbf{0}) \neq \mathbf{0}$, we can see that S is not a linear transformation.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation satisfying

$$\begin{aligned} T(\mathbf{e}_1) &= (1, 3) \\ T(\mathbf{e}_2) &= (4, -7) \\ T(\mathbf{e}_3) &= (-5, 4) \end{aligned}$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

i. $\boxed{\text{7 points}}$ Find the standard matrix of T.

Solution: To calculate a standard matrix, we must use the standard bases on $\mathbb{R}^2$ and $\mathbb{R}^3$. We find that

$$[T] = \big[[T(\mathbf{e}_1)] \quad [T(\mathbf{e}_3)] \quad [T(\mathbf{e}_3)]\big] = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}.$$

ii. $\boxed{\text{18 points}}$ Now let $\mathcal{A} = \{(1, 0), (1, 1)\}$ be a different basis for $\mathbb{R}^2$.
Find the matrix for $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the bases $\mathcal{B}$ and $\mathcal{A}$.

Solution: Since

$$\begin{aligned} T(\mathbf{e}_1) &= (1, 3) = -2(1, 0) + 3(1, 1) \\ T(\mathbf{e}_2) &= (4, -7) = 11(1, 0) - 7(1, 1) \\ T(\mathbf{e}_3) &= (-5, 4) = -9(1, 0) + 4(1, 1) \end{aligned}$$

we have $[T(\mathbf{e}_1)]_{\mathcal{A}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $[T(\mathbf{e}_2)]_{\mathcal{A}} = \begin{bmatrix} 11 \\ -7 \end{bmatrix}$ and $[T(\mathbf{e}_1)]_{\mathcal{A}} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. Hence

$$[T]_{\mathcal{A},\mathcal{B}} = \big[[T(\mathbf{e}_1)]_{\mathcal{A}} \quad [T(\mathbf{e}_3)]_{\mathcal{A}} \quad [T(\mathbf{e}_3)]_{\mathcal{A}}\big] = \begin{bmatrix} -2 & 11 & -9 \\ 3 & -7 & 4 \end{bmatrix}.$$