FORENAME:

SURNAME:

STUDENT NO:

DEPARTMENT:

TEACHER: ☐ Neil Course ☐ Vasfi Eldem ☐ Mehmet Kavuk ☐ Hasan Özekes

SIGNATURE:

- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, cell phones are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place [a box around your answer] to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Time limit is 80 min.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. [25 points] Calculate $\begin{vmatrix} 1 & 5 & 4 & 3 & 2 \\ 0 & 8 & 5 & 9 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 3 & 9 & 6 & 5 & 4 \\ 0 & 8 & 0 & 6 & 0 \end{vmatrix}$.

Solution:

$$\begin{vmatrix} 1 & 5 & 4 & 3 & 2 \\ 0 & 8 & 5 & 9 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 3 & 9 & 6 & 5 & 4 \\ 0 & 8 & 0 & 6 & 0 \end{vmatrix} = (-7)\begin{vmatrix} 1 & 4 & 3 & 2 \\ 0 & 5 & 9 & 0 \\ 3 & 6 & 5 & 4 \\ 0 & 0 & 6 & 0 \end{vmatrix} = (-7)(-6)\begin{vmatrix} 1 & 4 & 2 \\ 0 & 5 & 0 \\ 3 & 6 & 4 \end{vmatrix} = (-7)(-6)(+5)\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= (-7)(-6)(+5)(4-6) = -420.$$

2. 25 points Find the inverse of $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 + 3R_1 \to R_2$ and $R_3 - 2R_1 \to R_3$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$R_3 + 3R_2 \to R_3$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$\frac{1}{2}R_3 \to R_3$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$R_2 + 2R_3 \to R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$R_1 + 2R_3 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

3. 25 points Consider the following linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & & & - & 2x_4 & = & -3 \\ & & 2x_2 & + & 2x_3 & & & = & 0 \\ & & & & x_3 & + & 3x_4 & = & 1 \\ -2x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & = & 5 \end{array}$$

a. Write an augmented matrix for this linear system.

b. Put your augmented matrix into reduced row echelon form (RREF).

c. Find all the solutions (if any) of this linear system. If the linear system does not have any solutions, then you must explain why.

Solution:

a.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}$$

b. $R_4 + 2R_1 \to R_4$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$

$\frac{1}{2}R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$

$R_4 - 3R_2 \to R_4$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix}$$

$R_4 + R_3 \to R_4$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 - R_3 \to R_2$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c. The solution is

$$\begin{aligned} x_1 &= 2s - 3 \\ x_2 &= 3s - 1 \\ x_3 &= 1 - 3s \\ x_4 &= s \end{aligned}$$

for all $s \in \mathbb{R}$.

4. 25 points a. If A, B and C are $n \times n$ invertible matrices, does the equation

$$C^{-1}(A+X)B^{-1} = I_n$$

have a solution, X? If so, find it.

b. Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Find $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$ and $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$.

Solution:

a. Yes, this equation does have a solution, X. We calculate that

$$\begin{aligned} C^{-1}(A+X)B^{-1} &= I_n \\ CC^{-1}(A+X)B^{-1} &= CI_n \\ I_n(A+X)B^{-1} &= C \\ (A+X)B^{-1} &= C \\ (A+X)B^{-1}B &= CB \\ (A+X)I_n &= CB \\ A+X &= CB \\ X &= CB-A \end{aligned}$$

b. First

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -7$$

because swapping two rows multiplies the determinant by -1.

Multiplying a row by a constant k, also multiplies the determinant by k. So

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 14.$$

Since adding one row to another does not change the determinant, we have

$$\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} = 14$$

also.