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DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	25	
2	20	
3	30	
4	25	
Total:	100	

- The time limit is 90 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. (a) $\boxed{\text{10 points}}$ Determine whether the set $W = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$ forms a basis for $\mathbb{R}^4$.

Solution: Let us determine whether W is a linearly independent set.

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -4 & -3 \\ -3 & 2 & 1 & -8 \\ 2 & -3 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -4 & -3 \\ 0 & 2 & -8 & -5 \\ 0 & -3 & 12 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

W is a linearly dependent set, so it does not form a basis for $\mathbb{R}^4$.

(b) $\boxed{\text{15 points}}$ If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, then calculate $\begin{vmatrix} g & h & i \\ -3d+2a & -3e+2b & -3f+2c \\ 2a & 2b & 2c \end{vmatrix}$.

Solution:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3 \Rightarrow \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = -3 \Rightarrow \begin{vmatrix} g & h & i \\ -3d & -3e & -3f \\ 2a & 2b & 2c \end{vmatrix} = (-3)(2)(-3) \Rightarrow \begin{vmatrix} g & h & i \\ -3d+2a & -3e+2b & -3f+2c \\ 2a & 2b & 2c \end{vmatrix} = 18$$

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a-b \\ 2a+b-2c+d \\ 6b-4c-2d \end{bmatrix}$.

(a) $\boxed{\text{5 points}}$ Write the matrix representation of T.

Solution:

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a-b \\ 2a+b-2c+d \\ 6b-4c-2d \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & 1 & -2 & 1 \\ 0 & 6 & -4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(b) $\boxed{\text{7 points}}$ Find a basis for the kernel of T.

Solution:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 2 & 1 & -2 & 1 & 0 \\ 0 & 6 & -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 1 & 0 \\ 0 & 6 & -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}$$

$$x_4 = 0, \qquad 3x_2 - 2x_3 + x_4 = 0, \qquad x_1 - x_2 = 0 \Rightarrow \mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \end{bmatrix} x_2$$

(c) $\boxed{\text{5 points}}$ Find a basis for the image of T.

Solution: The first, second and fourth columns of A forms a basis for Im A

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \right\}$$

(d) $\boxed{\text{3 points}}$ Determine the rank T.

Solution: Rank T =3

3. (a) $\boxed{\text{15 points}}$ Find the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}$.

Solution:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 3 \\ -3 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)^2 + 9 = 0 \Rightarrow \lambda_1 = 4+3i, \lambda_2 = 4-3i$$

$$(A - (4+3i)I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 4-(4+3i) & 3 \\ -3 & 4-(4+3i) \end{bmatrix} = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} \sim \begin{bmatrix} -3i & 3 \\ -3i & 3 \end{bmatrix} \sim \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The eigenvector corresponding to $\lambda_2 = 4 - 3i$ is $\bar{\mathbf{v}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

(b) $\boxed{\text{15 points}}$ Find the matrix A^k where $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$.

Solution: Let us use $A^k = PD^kP^{-1}$ to find the result. The eigenvalues of the matrix A are $\lambda_1 = 5$, $\lambda_2 = 1$.

$$(A - 5I)\mathbf{v} = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \end{bmatrix} \Rightarrow \mathbf{v} = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$(A - I)\mathbf{w} = \mathbf{0} \Rightarrow \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} \Rightarrow \mathbf{w} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$$A^k = PD^kP^{-1} \Rightarrow A^k = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

4. (a) $\boxed{\text{10 points}}$ Find the corresponding eigenvectors of $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ with the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = \lambda_3 = 7$.

Solution:

$$\lambda_1 = -2 \Rightarrow (A + 2I)\mathbf{v_1} = \mathbf{0} \Rightarrow \begin{bmatrix} 5 & -2 & 4 & 0 \\ -2 & 8 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 & 0 \\ -2 & 8 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\lambda_1 = 7 \Rightarrow (A - 7I)\mathbf{v_2} = \mathbf{0} \Rightarrow \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v_2} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

The eigenvectors corresponding to $\lambda_1 = -2$ and $\lambda_2 = \lambda_3 = 7$ are $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, respectively.

(b) $\boxed{\text{15 points}}$ Find the matrix Q that orthogonally diagonalises A such that $A = QDQ^T$

Solution: Let us find orthonormal eigenvectors of A by using Gram Schmidt Process.

$$\mathbf{u_1} = \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathbf{u_2} = \mathbf{v_2} - \frac{<\mathbf{v_2}, \mathbf{u_1}>}{<\mathbf{u_1}, \mathbf{u_1}>}\mathbf{u_1} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \frac{0}{9}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{u_3} = \mathbf{v_3} - \frac{<\mathbf{v_3}, \mathbf{u_1}>}{<\mathbf{u_1}, \mathbf{u_1}>}\mathbf{u_1} - \frac{<\mathbf{v_3}, \mathbf{u_2}>}{<\mathbf{u_2}, \mathbf{u_2}>}\mathbf{u_2} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{0}{9}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} - \frac{-4}{5}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$\mathbf{u_1} = \frac{\mathbf{u_1}}{||\mathbf{u_1}||} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathbf{u_2} = \frac{\mathbf{u_2}}{||\mathbf{u_2}||} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\mathbf{u_3} = \frac{\mathbf{u_3}}{||\mathbf{u_3}||} = \frac{5}{3\sqrt{5}}\frac{1}{5}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \frac{1}{3\sqrt{5}}\begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \text{ and } Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{-2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$