FORENAME:

SURNAME:

STUDENT NO:

DEPARTMENT:

TEACHER: ☐ Neil Course ☐ Vasfi Eldem ☐ M.Tuba Gülpınar ☐ Hasan Özekes

SIGNATURE:

Question	Points	Score
1	20	
2	25	
3	25	
4	30	
Total:	100	

- The time limit is 70 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. $\boxed{\text{20 points}}$ Choose h, and k such that and the system has (a) no solution, (b) a unique solution, and (c) infinitely many solutions where $h \neq 0$.

$$\begin{aligned} hx_1 + kx_3 &= 2 \\ hx_1 + hx_2 + 4x_3 &= 4 \\ hx_2 + 2x_3 &= h \end{aligned}$$

Solution:

$$\begin{bmatrix} h & 0 & k & 2 \\ h & h & 4 & 4 \\ 0 & h & 2 & h \end{bmatrix} \sim \begin{bmatrix} h & 0 & k & 2 \\ 0 & h & 4-k & 2 \\ 0 & h & 2 & h \end{bmatrix} \sim \begin{bmatrix} h & 0 & k & 2 \\ 0 & h & 4-k & 2 \\ 0 & 0 & k-2 & h-2 \end{bmatrix}$$

If $h \neq 2$ and $k = 2$, then the system has no solution. $\begin{bmatrix} h & 0 & 2 & 2 \\ 0 & h & 2 & 2 \\ 0 & 0 & 0 & h-2 \end{bmatrix}$

If $k \neq 2$, then the system has a unique solution. $\begin{bmatrix} h & 0 & k & 2 \\ 0 & h & 4-k & 2 \\ 0 & 0 & k-2 & h-2 \end{bmatrix}$

If $h = 2$ and $k = 2$, then the system has infinitely many solutions. $\begin{bmatrix} 2 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. (a) $\boxed{\text{10 points}}$ Let $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$. Determine whether H is a subspace of $\mathbb{R}^3$.

Solution: Let c is a scalar and $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in H$.

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in H \Rightarrow x + y + z = 1$$

$$c\mathbf{u} = c \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \Rightarrow cx + cy + cz = c(x + y + z) = c \Rightarrow c\mathbf{u} \notin H$$

H is not a subspace of $\mathbb{R}^3$.

(b) $\boxed{\text{15 points}}$ Use cofactor expansion to compute the following determinant.

$$\begin{vmatrix} 4 & 0 & -6 & 4 & -6 \\ 0 & 0 & 3 & 0 & 0 \\ 8 & 3 & -7 & 5 & -7 \\ 0 & 0 & 5 & 2 & -2 \\ 0 & 0 & 9 & -1 & 4 \end{vmatrix}$$

Solution:

$$\begin{aligned} \begin{vmatrix} 4 & 0 & -6 & 4 & -6 \\ 0 & 0 & 3 & 0 & 0 \\ 8 & 3 & -7 & 5 & -7 \\ 0 & 0 & 5 & 2 & -2 \\ 0 & 0 & 9 & -1 & 4 \end{vmatrix} &= 3(-1)^{2+3} \begin{vmatrix} 4 & 0 & 4 & -6 \\ 8 & 3 & 5 & -7 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 4 \end{vmatrix} \\ &= (-3) \left[3(-1)^{2+2} \begin{vmatrix} 4 & 4 & -6 \\ 0 & 2 & -2 \\ 0 & -1 & 4 \end{vmatrix} \right] \\ &= (-9) \left[4(-1)^{1+1} \begin{vmatrix} 2 & -2 \\ -1 & 4 \end{vmatrix} \right] \\ &= (-9)(24) = -216 \end{aligned}$$

3. (a) $\boxed{15 \text{ points}}$ Do the columns of the matrix $A = \begin{bmatrix} 1 & 4 & -2 & 0 & 1 \\ 1 & 4 & -2 & -2 & 0 \\ -3 & -6 & 8 & 5 & 0 \\ 2 & 2 & -6 & 1 & 2 \end{bmatrix}$ form a linearly dependent set? If the answer is yes, find the relationship between them.

Solution:

$$A = \begin{bmatrix} 1 & 4 & -2 & 0 & 1 \\ 1 & 4 & -2 & -2 & 0 \\ -3 & -6 & 8 & 5 & 0 \\ 2 & 2 & -6 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 6 & 2 & 5 & 3 \\ 0 & -6 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 6 & 2 & 5 & 3 \\ 0 & 0 & 0 & 6 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -2 & 0 & 1 \\ 0 & 6 & 2 & 5 & 3 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -2 & 0 & 1 \\ 0 & 6 & 2 & 5 & 3 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2c_4 + c_5 = 0 \Rightarrow c_4 = -\frac{1}{2}c_5$$

$$6c_2 + 2c_3 + 5c_4 + 3c_5 = 0 \Rightarrow c_2 = \frac{1}{6}(-2c_3 - 5c_4 - 3c_5) = -\frac{1}{3}c_3 - \frac{1}{12}c_5$$

$$c_1 + 4c_2 - 2c_3 + c_5 = 0 \Rightarrow c1 = -4c_2 + 2c_3 - c5 \Rightarrow c_1 = \frac{10}{3}c_3 - \frac{2}{3}c_5$$

The columns of A form a linear dependent set. Let us find the relationship between them. Let us take $c_3 = 3$ and $c_5 = 12$, then $c_1 = 2$, $c_2 = -2$ and $c_4 = -6$. Thefore,

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{bmatrix}$$

$$2\mathbf{a}_1 - 2\mathbf{a}_2 + 3\mathbf{a}_3 - 6\mathbf{a}_4 + 12\mathbf{a}_5 = 0$$

(b) $\boxed{10 \text{ points}}$ Compute $\det(2A^3)$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 1 & 3 & 3 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 1 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -5$$

$$\det(2A^3) = 2^3(\det A)^3 = 8(-5)^3 = -1000$$

4. (a) $\boxed{\text{15 points}}$ Find the inverse of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 1 \\ -3 & 5 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -3 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ -3 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & 1 & 0 \\ 0 & 5 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & -4 & -\frac{2}{3} & -5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{5}{4} & -\frac{1}{4} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{5}{4} & -\frac{1}{4} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{5}{4} & -\frac{1}{4} \end{bmatrix}$$

(b) $\boxed{\text{15 points}}$ <u>Use Cramer's Rule</u> to find the value of x_3 for the system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 2 & -2 & 1 & -1 \\ 1 & 4 & 2 & 2 \\ 0 & -1 & 2 & 0 \\ 1 & 3 & 2 & 2 \end{bmatrix}$ with $\det A = 10$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$.

Solution:

$$x_3 = \frac{\det A_3}{\det A} = \frac{\begin{vmatrix} 2 & -2 & -1 & -1 \\ 1 & 4 & 3 & 2 \\ 0 & -1 & 0 & 0 \\ 1 & 3 & 2 & 2 \end{vmatrix}}{10} = \frac{(-1)(-1)^{3+2}\begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{vmatrix}}{10} = \frac{\begin{vmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{vmatrix}}{10} = \frac{\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}}{10} = \frac{5}{10} = 0.5$$