FORENAME:

SURNAME:

STUDENT NO:

DEPARTMENT:

TEACHER: ☐ Neil Course ☐ Vasfi Eldem ☐ M.Tuba Gülpınar ☐ Hasan Özekes

SIGNATURE:

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- The time limit is 70 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. $\boxed{\text{25 points}}$ Find a basis for the space spanned by the vectors $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}6\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}0\\3\\-1\\1\end{bmatrix}\right\}$.

Solution:

$$A = \begin{bmatrix}1 & -2 & 6 & 5 & 0\\0 & 1 & -1 & -3 & 3\\0 & -1 & 2 & 3 & -1\\1 & 1 & -1 & -4 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 6 & 5 & 0\\0 & 1 & -1 & -3 & 3\\0 & -1 & 2 & 3 & -1\\0 & 3 & -7 & -9 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 6 & 5 & 0\\0 & 1 & -1 & -3 & 3\\0 & 0 & 1 & 0 & 2\\0 & 0 & -4 & 0 & -8\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 6 & 5 & 0\\0 & 1 & -1 & -3 & 3\\0 & 0 & 1 & 0 & 2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

The first, second, and third columns of A are linearly independent. Therefore,

$$\text{Span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}6\\-1\\2\\-1\end{bmatrix}, \begin{bmatrix}5\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}0\\3\\-1\\1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}6\\-1\\2\\-1\end{bmatrix}\right\}$$

2. Let $A = \begin{bmatrix} 1 & 2 & 3 & 6 & -3 & 5 \\ 0 & 5 & 5 & 6 & -1 & 10 \\ -2 & 0 & -2 & 3 & -5 & -2 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ be row equivalent matrices.

(a) $\boxed{\text{10 points}}$ Find a basis for Nul A .

Solution: The system has infinitely many solutions depend on 6-3=3 parameters.

$$\begin{array}{c} x_4 - x_5 = 0 \Rightarrow x_4 = x_5 \\ x_2 + x_3 + x_5 + 2x_6 = 0 \Rightarrow x_2 = -x_3 - x_5 - 2x_6 \\ x_1 + x_3 + x_5 + x_6 = 0 \Rightarrow x_1 = -x_3 - x_5 - x_6 \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 - x_6 \\ -x_3 - x_5 - 2x_6 \\ x_3 \\ x_5 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_6$$

The set $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

(b) $\boxed{\text{7 points}}$ Find a basis for Col A .

Solution: The first, second and fourth columns of B involve pivots, so the first, second , and fourth columns of A form a basis for Col A. Hence, $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 6 \\ 3 \\ 0 \end{bmatrix} \right\}$ forms a basis for Col A

(c) $\boxed{\text{8 points}}$ Find the rank and nullity of A .

Solution: Rank A= dim (Col A)= 3
Nullity = dim (Nul A)=3

3. Let $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \right\}$ and $W = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \right\}$ be two bases for $\mathbb{R}^3$.

(a) $\boxed{\text{10 points}}$ Find the coordinates of $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ relative to the basis V.

Solution:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 2 & 1 \\ 1 & -3 & -3 \\ 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 2 \\ 1 & -3 & -3 & -1 \\ 0 & -3 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & -3 & -5 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$[\mathbf{v}]_V = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

(b) $\boxed{\text{10 points}}$ Find the change of coordinates matrix $\underset{W \leftarrow V}{P}$ from V to W.

Solution: We can use $\underset{W\leftarrow V}{P} = W^{-1}V$, $[W|V] \sim \left[I|\underset{W\leftarrow V}{P}\right]$ or $\underset{W\leftarrow V}{P} = \left[[\mathbf{v}_1]_W \quad [\mathbf{v}_2]_W \quad [\mathbf{v}_3]_W\right]$

$$[W|V] \sim \left[I|\underset{W\leftarrow V}{P}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 2 & 2 & 3 & 1 & -3 & -3 \\ 4 & 3 & 6 & 0 & -3 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & -1 & 3 & -7 & -5 \\ 0 & -1 & -2 & 4 & -11 & -9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -4 & 11 & 9 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -9 & -8 \\ 0 & 1 & 2 & -4 & 11 & 9 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -9 & -8 \\ 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & -3 & 7 & 5 \end{array}\right]$$

$$\underset{W\leftarrow V}{P} = \begin{bmatrix} 3 & -9 & -8 \\ 2 & -3 & -1 \\ -3 & 7 & 5 \end{bmatrix}$$

(c) $\boxed{\text{5 points}}$ Find the coordinates of $\mathbf{v}$ relative to W by using $\underset{W\leftarrow V}{P}$.

Solution:

$$[\mathbf{v}]_W = \underset{W\leftarrow V}{P}[\mathbf{v}]_V = \begin{bmatrix} 3 & -9 & -8 \\ 2 & -3 & -1 \\ -3 & 7 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 5 \end{bmatrix}$$

4. (a) $\boxed{\text{10 points}}$ Let $L : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y+3z \\ x+y-2z \\ 4x+y \\ 3x-y-z \end{bmatrix}$. Find the matrix representation of L.

Solution:

$$L(\mathbf{x}) = A\mathbf{x}$$

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y+3z \\ x+y-2z \\ 4x+y \\ 3x-y-z \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) $\boxed{\text{15 points}}$ Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. Find a polynomial $\mathbf{p} \in \mathbb{P}_2$ which is a basis for kernel of T.

Solution:

$$\ker T = \{\mathbf{p} : \mathbf{p} \in \mathbb{P}_2 \text{ and } T(\mathbf{p}) = \mathbf{0}\}$$

$$\mathbf{p}(t) = a + bt + ct^2 \Rightarrow T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = \begin{bmatrix} a \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} a = 0 \\ b + c = 0 \end{matrix} \Rightarrow \mathbf{p}(t) = -ct + ct^2$$

$$\ker T = \left\{\mathbf{p} : \mathbf{p}(t) = \left(-t + t^2\right) c, c \in \mathbb{R}\right\} = \text{Span}\left\{-t + t^2\right\}$$

$$\mathbf{p}(t) = -t + t^2$$