FORENAME:

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STUDENT NO:

DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- The time limit is 75 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. (a) $\boxed{\text{10 points}}$ Find the inverse of the matrix $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1 & -1/2 \end{array}\right]$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1 & -1/2 \end{array}\right] \Rightarrow B^{-1} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix}$$

(b) $\boxed{\text{15 points}}$ Compute $\det B^4$, where $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ **by using cofactor expansion or elemantary row operations.**

Solution:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{vmatrix} = 2(-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = (-2)$$

$$\det B^4 = (\det B)^4 = (-2)^4 = 16$$

2. Let $T : P_3 \to P_2$ be a linear transformation where $T(ax^3 + bx^2 + cx + d) = (a - b - c)x^2 + (2a - d)x + (b + c + d)$.
   (a) $\boxed{\text{10 points}}$ Find the standard matrix (matrix representation) of the linear transformation T.
   (b) $\boxed{\text{15 points}}$ Find a basis for the kernel of T.

Solution:

(a)

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a-b-c \\ 2a-d \\ b+c+d \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{c} x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \Rightarrow x_2 = -x_3 \\ x_1 - x_2 - x_3 = 0 \Rightarrow x_1 = 0 \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} x_3$$

$\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$ forms a basis for kernel of T.

3. 25 points Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for the vector space V, and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

(a) Find the change of coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

Solution:

$$[\mathbf{a}_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad [\mathbf{a}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad [\mathbf{a}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

(b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

Solution:

$$[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

4. $\boxed{\text{25 points}}$ Let $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ be a matrix whose eigenvalues are $\lambda_1 = \lambda_2 = 3,\ \lambda_3 = 6$.

(a) Find the corresponding eigenvectors of A.

Solution:

$$(A-3I)\mathbf{v} = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$(A-6I)\mathbf{w} = 0 \Rightarrow \begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) Find the matrices P and D that orthogonally diagonalize the matrix A such that $A = PDP^T$ where $P^T = P^{-1}$.

Solution: We must obtain orthonormal eigenvectors of A.

$$\mathbf{u}_1 = \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{<\mathbf{v}_2, \mathbf{u}_1>}{<\mathbf{u}_1, \mathbf{u}_1>}\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1/\sqrt{2}}{1} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} \Rightarrow \mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{||\mathbf{w}||}\mathbf{w} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$