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DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- The time limit is 75 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. $\boxed{\text{25 points}}$ For which values of k does the system $\begin{aligned} x+2y+6z&=2\\ y+2kz&=0\\ kx+2z&=1\end{aligned}$ have

(a) no solution.

(b) infinitely many solutions.

(c) a unique solution.

Solution: Let us transform the augmented matrix to row echelon form.

$$\begin{bmatrix}1&2&6&2\\0&1&2k&0\\k&0&2&1\end{bmatrix}\sim\begin{bmatrix}1&2&6&2\\0&1&2k&0\\0&-2k&2-6k&1-2k\end{bmatrix}\sim\begin{bmatrix}1&2&6&2\\0&1&2k&0\\0&0&4k^2-6k+2&1-2k\end{bmatrix}\sim\begin{bmatrix}1&2&6&2\\0&1&2k&0\\0&0&2(2k-1)(k-1)&1-2k\end{bmatrix}$$

If $k=1$, then we obtain $\begin{bmatrix}1&2&6&2\\0&1&2&0\\0&0&0&-1\end{bmatrix}$. Therefore, the system has no solution.

If $k=\frac{1}{2}$, then we obtain $\begin{bmatrix}1&2&6&2\\0&1&1&0\\0&0&0&0\end{bmatrix}$. Therefore, the system has infinitely many solutions.

If $k\neq 1$ and $k\ \neq\frac{1}{2}$, then we obtain $\begin{bmatrix}1&2&6&2\\0&1&2k&0\\0&0&1&-\dfrac{1}{2(k-1)}\end{bmatrix}$ Therefore, the system have a unique solution.

2. (a) $\boxed{\text{15 points}}$ Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 4 & 0 & 4 & 0 \\ 2 & 0 & 1 & 1 \end{vmatrix} = 2(-1)^{1+2} \begin{vmatrix} 0 & 4 & -2 \\ 4 & 4 & 0 \\ 2 & 1 & 1 \end{vmatrix} + 0 + 0 + 0$$

$$= (-2) \begin{vmatrix} 0 & 4 & -2 \\ 0 & 2 & -2 \\ 2 & 1 & 1 \end{vmatrix} = (-2) \left[(2)(-1)^{3+1} \begin{vmatrix} 4 & -2 \\ 2 & -2 \end{vmatrix} \right] = (-4)(-8+4) = 16$$

(b) $\boxed{\text{10 points}}$ Suppose that $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 0 & 2 & 4 & -2 \\ 3 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ and $\det A = (-2)$. Use Cramer's rule to find x_2.

Solution:

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 4 & -2 \\ 3 & -2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{vmatrix}}{-2} = \frac{16}{-2} = -8$$

3. (a) $\boxed{\text{10 points}}$ Find the adjoint matrix $(Adj A)$ of $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 6 \\ 0 & 2 \end{vmatrix} = 2 \quad C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 6 \\ 0 & 2 \end{vmatrix} = -4 \quad C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 6 \\ 1 & 6 \end{vmatrix} = 6$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 6 \\ 0 & 2 \end{vmatrix} = 0 \quad C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 6 \\ 0 & 2 \end{vmatrix} = 2 \quad C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 6 \\ 0 & 6 \end{vmatrix} = -6$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0 \quad C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$Adj A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 6 & -6 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\boxed{\text{15 points}}$ Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution: First Way:

$$A^{-1} = \frac{1}{\det A} Adj A = \frac{1}{2} \begin{bmatrix} 2 & -4 & 6 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Second Way:

$$[A|I] \sim [I|A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 1 & -2 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0.5 \end{array}\right]$$

4. (a) $\boxed{\text{10 points}}$ Suppose an $n \times n$ matrix A satisfies the equation $A^2 - 2A + I = 0$. Show that $A^3 = 3A - 2I$.

Solution:

$$A^2 = 2A - I \Rightarrow A^3 = 2A^2 - A = 2(2A - I) - A = 3A - 2I$$

(b) $\boxed{\text{15 points}}$ Let A, B and C be 3×3 matrices with $\det A = -3$, $\det B = 4$ and $\det C = 2$. Compute $\det(2A^2B^{-2}C^T)$.

Solution:

$$\det(2A^2B^{-2}C^T) = 2^3(\det A)^2 \frac{1}{(\det B)^2}(\det C) = 8(-3)^2 \frac{1}{4^2} 2 = 9$$