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DEPARTMENT:

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SIGNATURE:

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- The time limit is 75 minutes.
- Any attempts at cheating or plagiarizing and assisting of such actions in any form would result in getting an automatic zero (0) from the exam. Disciplinary action will also be taken in accordance with the regulations of the Council of Higher Education.
- Give your answers in exact form (for example $\frac{\pi}{3}$ or $5\sqrt{3}$), except as noted in particular problems.
- Calculators, mobile phones, smart watches, etc. are not allowed.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- Use a **BLUE ball-point pen** to fill the cover sheet. Please make sure that your exam is complete.
- Do not write in the table above.

1. (a) $\boxed{\text{15 points}}$ Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$. Is W a subspace of $\mathbb{R}^2$? Why?

Solution: Let us take $\mathbf{u} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \in W$.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$
$$(-2)(2) \leq 0 \Rightarrow \mathbf{u} + \mathbf{v} \notin W$$

W is not a subspace of $\mathbb{R}^2$

(b) $\boxed{\text{10 points}}$ Give an example of a 5-dimensional vector space and write down a basis of it.

Solution: $\mathbb{R}^5$ is a 5-dimensional space and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of it.

2. (a) $\boxed{\text{15 points}}$ Let T be a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$. Find a vector $\mathbf{x}$ whose image under T is $\mathbf{b} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, and determine whether $\mathbf{x}$ is unique.

Solution: Let us solve the system$A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

$$x_3 \text{is free } \Rightarrow x_2 = \frac{1}{2} - 2x_3$$

$$x_1 = -2 + 5x_2 + 7x_3 = -2 + 5(\frac{1}{2} - 2x_3) + 7x_3 = \frac{1}{2} - 3x_3$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 3x_3 \\ \frac{1}{2} - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} x_3$$

The system has infinitely many solutions depend on on parameter. For example, we may take $\mathbf{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$

(b) $\boxed{\text{10 points}}$ Assume that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, and $T(\mathbf{e}_3) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$, where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the columns of the 3×3 identity matrix. Find the standard matrix of T.

Solution:

$$T = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$$

3. (a) $\boxed{\text{10 points}}$ Find a matrix A such that $\text{Col}A = \left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r,s,t \text{ real} \right\}$.

Solution:

$$\begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} r + \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} s + \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} t \Rightarrow A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$$

(b) $\boxed{\text{15 points}}$ The set $\mathcal{B} = \{1-t^2, t-t^2, 2-2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3+t-6t^2$ relative to $\mathcal{B}$.

Solution:

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow 3+t-6t^2 = c_1(1-t^2) + c_2(t-t^2) + c_3(2-2t+t^2)$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$c_3 = -2, \quad c_2 - 2c_3 = 1 \Rightarrow c_2 = -3, \quad c_1 + 2c_3 = 3 \Rightarrow c_1 = 7$$

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$$

4. $\boxed{\text{25 points}}$ Let $A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find a basis for the null space of A

Solution: Let us solve the system $A\mathbf{x} = \mathbf{0}$.

$$x_4 - 2x_5 - 3x_3 = 0 \Rightarrow x_4 = 2x_5 + 3x_6$$
$$x_3 - 3x_4 + x_5 = 0 \Rightarrow x_3 = 3x_4 - x_5 = 5x_5 + 9x_6$$
$$x_1 + 3x_2 + 2x_4 - x_5 + x_6 \Rightarrow x_1 = -3x_2 - 2x_4 + x_5 - x_6 = -3x_2 - 3x_5 - 7x_6$$
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} -3x_2 - 3x_5 - 7x_6 \\ x_2 \\ 5x_5 + 9x_6 \\ 2x_5 + 3x_6 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 5 \\ 2 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} -7 \\ 0 \\ 9 \\ 3 \\ 0 \\ 1 \end{bmatrix} x_6$$

$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 9 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms a basis for Nul A.

(b) Find a basis for the column space of A

Solution: The pivot columns of A forms a basis for column space of A. Therefore, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ forms a basis for Col A.

(c) Determine the nullity and rank of A.

Solution: The nullity of the matris A is dim (Nul A)=3
The rank of the matrix A is dim (Col A)=3.