OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

05.01.2012 **MAT 371 – Diferansiyel Denklemler – Yarıyıl Sonu Sınavı** N. Course

ADI SOYADI
ÖĞRENCİ NO
İMZA

Do not open the exam until you are told that you may begin.
Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.

1. You will have 120 minutes to answer 4 questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You should write your student number on every page.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the final 10 minutes of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam 120 dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan 4 tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Öğrenci numaranızı her sayfaya yazınız.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

1	2	3	4	5	TOTAL

Question 1 (Second Order Linear Differential Equations)**.** Find the general solution $y : \mathbb{R} \to \mathbb{R}$ of

$$y'' + 4y' + 5y = 325e^t \sin 3t + t^2 - 2. \tag{1}$$

You will be given points for the following:

- [8 pts] Finding the general solution of the homogeneous equation $y'' + 4y' + 5y = 0$.
- [7 pts] Finding a particular solution to $y'' + 4y' + 5y = 325e^t \sin 3t$.
- [7 pts] Finding a particular solution to $y'' + 4y' + 5y = t^2 - 2$.
- [3 pts] Writing down the general solution of (1).

$$y'' + 4y' + 5y = 325e^t \sin 3t + t^2 - 2. \tag{1}$$

Therefore, the general solution of (1) is

$y(t) =$

Question 2 (Reduction of Order)**.**

(a) [2 pts] Consider

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0, \qquad x > 1. \tag{2}$$

Show that $y_1(x) = e^x$ is a solution of (2).

(b) [15 points] Using the method of reduction of order, find a second solution $y_2(x)$ of (2).
[HINT: Write $y_2(x) = v(x)y_1(x)$. Note that $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$ and $\int \frac{1}{x-1}\ dx = \log|x-1| + \mathit{constant}$.]

(c) [4 pts] Show that y_1 and y_2 are linearly independent.

(d) [4 pts] Solve

$$\begin{cases} (x-1)y'' - xy' + y = 0, \qquad x > 1 \\ y(2) = 100 \\ y'(2) = 50. \end{cases}$$

Question 3 (Separable Equations)**.** Consider the differential equation

$$\frac{dy}{dx} = \frac{y-4x}{x-y}. \tag{3}$$

(a) [2 pts] Show that this differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)-4}{1-\left(\frac{y}{x}\right)}. \tag{4}$$

(b) [2 pts] Let $v(x)$ be a new variable such that $v = y/x$ or $y(x) = xv(x)$. Differentiate $y = xv$ to find $\frac{dy}{dx}$ in terms of x, v, and $\frac{dv}{dx}$.

(c) [5 ps] By using (4), $y = xv$ and your answer to (b), show that

$$x\frac{dv}{dx} = \frac{v^2-4}{1-v}.$$

(d) [12 pts] The equation

$$x\frac{dv}{dx} = \frac{v^2-4}{1-v}.$$

is a separable differential equation. Solve this equation.

[HINT: $\frac{1-t}{t^2-4} = \frac{A}{(t-2)} + \frac{B}{(t+2)}$ for some $A, B \in \mathbb{R}$, and $\int \frac{1}{t+k}\, dt = \log|t+k| + constant$. You do not need to find an explicit solution; an implicit solution is good enough.]

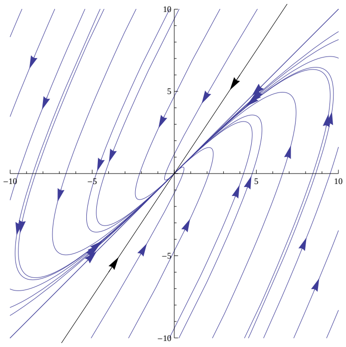
(e) [4 pts] Now use $y(x) = xv(x)$ to find the solution of (3).

Question 4 (Systems of Equations)**.**

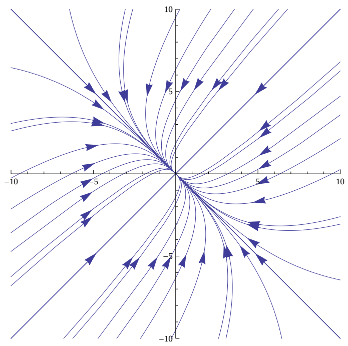
(a) [11 pts] Solve

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

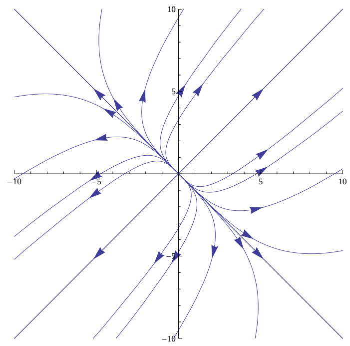
(b) [2 pts] How does the solution behave as $t \to \infty$?



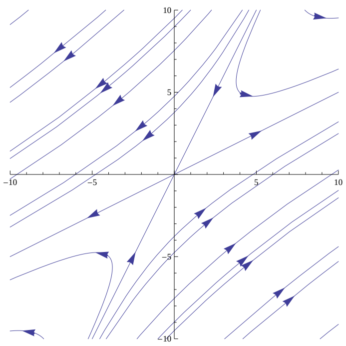
(i) Stable node



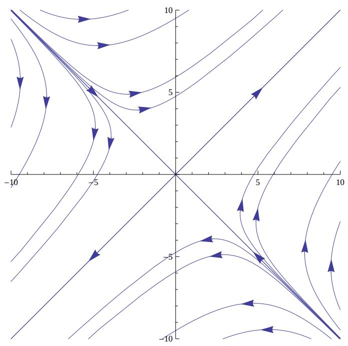
(ii) Stable node



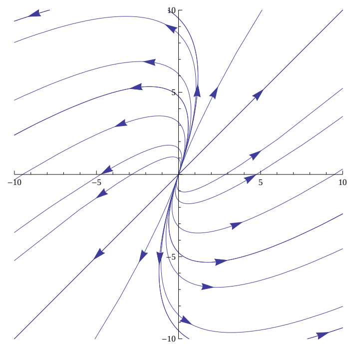
(iii) Unstable node



(iv) Saddle



(v) Saddle



(vi) Unstable node

Let $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$. The eigenvalues of A are $r_1 = -1$ and $r_2 = 2$. The corresponding eigenvectors are $\xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively.

(c) [2p] Which of the graphs (above) is the phase plot of the equation $\mathbf{x}' = A\mathbf{x}$?

[Mark ☑ only one box.]

☐ (i) ☐ (ii) ☐ (iii) ☐ (iv) ☐ (v) ☐ (vi)

(d) [10p] Justify (explain) your answer to part (c).

Question 5 (Convergence of Euler's Method)**.** Consider the initial value problem

$$\begin{cases} y' = 1 - t + y \\ y(t_0) = y_0. \end{cases} \tag{5}$$

(a) [3p] Show that the solution of (5) is

$$y = \phi(t) = (y_0 - t_0)e^{(t-t_0)} + t.$$

[HINT: Don't forget to check that this satisfies the initial condition as well as the differential equation.]

Recall that Euler's formula for the numerical approximation of the first order differential equation $\frac{dy}{dt} = f(t, y)$ is

$$y_{n+1} = y_n + f_n h$$

where $t_{n+1} = t_n + h$ and $f_n = f(t_n, y_n)$.

(b) [2p] Using Euler's formula and (5), show that

$$y_k = (1 + h)y_{k-1} + h - ht_{k-1}, \qquad k = 1, 2, 3, \ldots.$$

[HINT: Our f is, of course, $f(t, y) = 1 - t + y$.]

$$y_k = (1+h)y_{k-1} + h - ht_{k-1}, \qquad k = 1, 2, 3, \ldots.$$

(c) [2p] Show that

$$y_1 = (1+h)(y_0 - t_0) + t_1.$$

(d) [9p] Use *proof by induction* to show that

$$y_n = (1+h)^n (y_0 - t_0) + t_n \tag{6}$$

for all $n \in \mathbb{N}$.

(e) [9p] Now let $t > t_0$ be fixed. For each $n \in \mathbb{N}$, choose $h = h(n) = \frac{t-t_0}{n}$. Note that $t_n = t_0 + nh = t_0 + n\left(\frac{t-t_0}{n}\right) = t$ for all n. Note also that $h \to 0$ as $n \to \infty$.

Use (6) to show that $y_n \to \phi(t)$ as $n \to \infty$.

[HINT: $\lim_{n\to\infty}(1+\frac{a}{n})^2 = e^a$.]