Question 1 (Exact Equations). Consider

$$(1 - y - e^x) + \frac{dy}{dx} = 0 \tag{1}$$

This equation is of the form $M(x, y) + N(x, y)y' = 0$.

(a) [5 points] Is this equation exact?

$M(x,y) = 1 - y - e^x$. $M_y = -1$. $N = 1$. $N_x = 0$.

No, the equation is not exact.

(b) [5 points] Calculate $\frac{M_y - N_x}{N}$ and $\frac{N_x - M_y}{M}$.

$$\frac{M_y - N_x}{N} = \frac{-1-0}{1} = -1$$

$$\frac{N_x - M_y}{M} = \frac{0-(-1)}{1-y-e^x} = \frac{1}{1-y-e^x}.$$

(c) [10 points] If $\left(\frac{M_y - N_x}{N}\right) = P(x)$ is a function only of x (i.e. there is no y), then find an integrating factor $\mu(x)$ that solves

$$\frac{d\mu}{dx}(x) = \left(\frac{M_y - N_x}{N}\right)\mu(x);$$

OR if $\left(\frac{N_x - M_y}{M}\right) = Q(y)$ is a function only of y (i.e. there is no x), then find an integrating factor $\mu(y)$ that solves

$$\frac{d\mu}{dy}(y) = \left(\frac{N_x - M_y}{M}\right)\mu(y).$$

$\frac{d\mu}{dx} = (-1)\mu$. So $\frac{d\mu}{\mu} = -dx$.

So $\log|\mu| = -x + c$. So $\mu = Ce^{-x}$. Choose $C = 1$.

$\mu(x) = e^{-x}$.

$\mu(x) = e^{-x}$

$$(1 - y - e^x) + \frac{dy}{dx} = 0 \qquad (1)$$

(d) [5 points] Multiply equation (1) by the integrating factor that you found in part (c). Is the equation now exact?

$$\left(e^{-x} - ye^{-x} - 1\right) + \left(e^{-x}\right)\frac{dy}{dx} = 0.$$

Now $M = e^{-x} - ye^{-x} - 1$, $M_y = -e^{-x}$, $N = e^{-x}$, $N_x = -e^{-x}$.

Yes, the equation is now exact.

(e) [20 points] Solve the equation that you wrote in part (d).

We need to find $\psi(x,y)$ such that $\psi_x = e^{-x} - ye^{-x} - 1$ and $\psi_y = e^{-x}$. Integrating ψ_x wrt x gives

$$\psi = -e^{-x} + ye^{-x} - x + h(y).$$

Then differentiating wrt y gives

$$\psi_y = e^{-x} + h'(y).$$

So we can choose $h(y) = 0$. The solution of (1) is

$$-e^{-x} + ye^{-x} - x = c.$$

So $y = ce^{x} + xe^{x} + 1$ } optional

(f) [5 points] Now find the explicit solution of

$$\begin{cases} (1 - y - e^x) + \frac{dy}{dx} = 0 \\ y(0) = 7. \end{cases}$$

$$-e^{-x} + ye^{-x} - x = c.$$

Setting $x = 0$ and $y = 7$, we get.

$$c = -e^0 + 7e^0 - 0 = -1 + 7 = 6.$$

So

$$e^{-x} + ye^{-x} - x = 6 \quad \left(\text{or } y = 6e^{x} + xe^{x} + 1\right).$$

optional.

Question 2 (Autonomous Equations). Consider the initial value problem

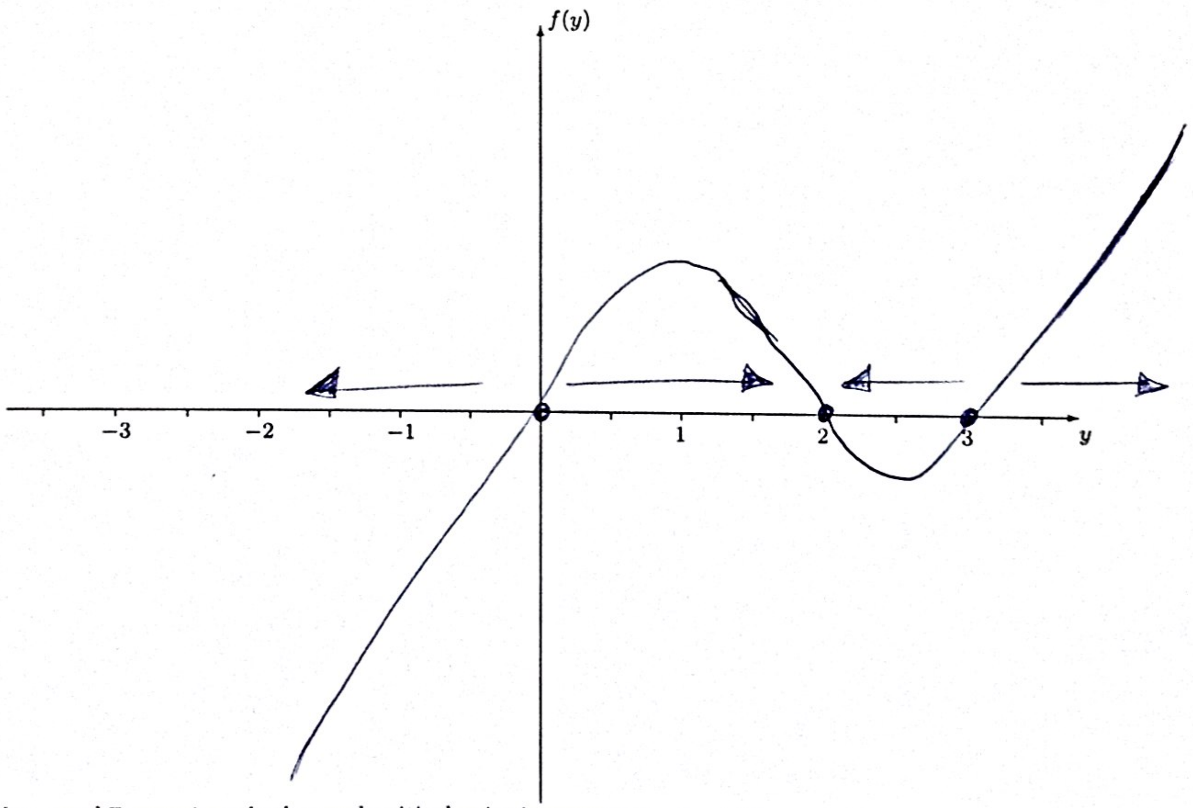
$$\begin{cases} \frac{dy}{dt} = f(y) = 4(1-\frac{y}{2})(1-\frac{y}{3})y \\ y(0) = y_0 \end{cases} \tag{2}$$

where $-\infty < y_0 < \infty$.

(a) [5 points] Find all of the critical points of the differential equation.

$f(y) = 0 \Rightarrow \quad y = 0, \ y = 2$ and $y = 3$.

(b) [12 points] Sketch the graph of $f(y)$ versus y.



(c) [6 points] Determine whether each critical point is asymptotically stable, unstable or semistable.

$y = 0$ is unstable

$y = 2$ is asymptotically stable

$y = 3$ is unstable.

$$\begin{cases} \frac{dy}{dt} = f(y) = 4(1 - \frac{y}{2})(1 - \frac{y}{3})y \\ y(0) = y_0 \end{cases} \qquad (2)$$

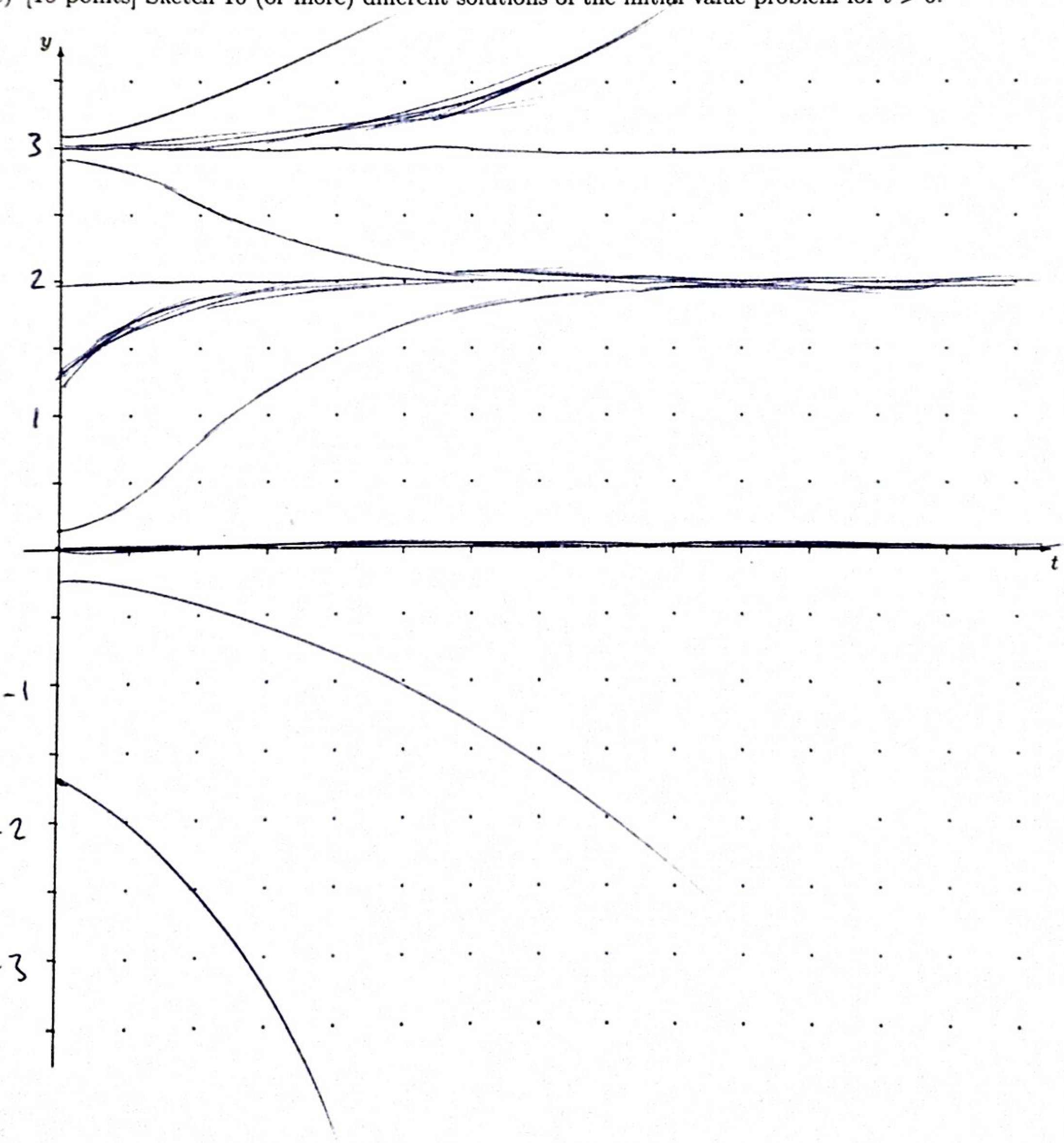
(d) [12 points] Determine where the graph of y versus t is concave up and where it is concave down. [Hint: First find the points where $f'(y) = 0$]

$$f(y) = 4\left(1 - \frac{y}{2} - \frac{y}{3} + \frac{y^2}{6}\right)y = 4\left(1 - \frac{5y}{6} + \frac{y^2}{6}\right)y$$
$$= 4\left(y - \frac{5}{6}y^2 + \frac{1}{6}y^3\right)$$
$$f'(y) = 4\left(1 - \frac{5}{3}y + \frac{1}{2}y^2\right) = 24(6 - 10y + 3y^2).$$
$$f'(y) = 0 \Rightarrow y = \frac{+10 \pm \sqrt{100 - 72}}{6} = \frac{1}{3}\left(5 \pm \sqrt{7}\right).$$

On $y \in (-\infty, 0)$, $f < 0$ and $f' > 0$. y is concave down
$(0, \frac{1}{3}(5-\sqrt{7}))$, $f > 0$ and $f' > 0$. y is concave up
$(\frac{1}{3}(5-\sqrt{7}), 2)$ $f > 0$ and $f' < 0$. y is concave down

On $(2, \frac{1}{3}(5+\sqrt{7}))$, $f < 0$, $f' < 0$. y is concave up
On $(\frac{1}{3}(5+\sqrt{7}), 3)$, $f < 0$, $f' > 0$. y is concave down
On $(3, \infty)$ $f > 0$, $f' > 0$. y is concave up.

(e) [15 points] Sketch 10 (or more) different solutions of the initial value problem for $t > 0$.

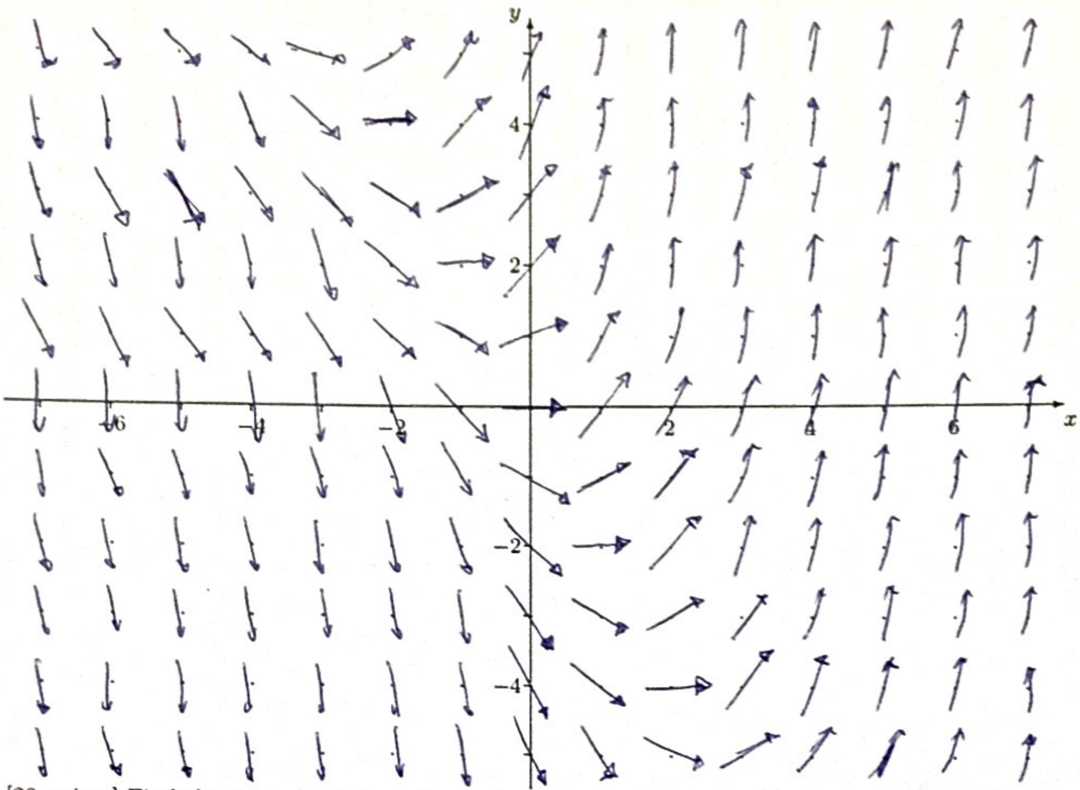


$$\frac{dy}{dt} = \frac{1}{2}y + t.$$

Question 3 (Linear Equations).

(a) [10 pts] Draw a direction field for

$$2\frac{dy}{dt} - y = 2t. \qquad (3)$$



(b) [20 points] Find the general solution of

$$2\frac{dy}{dt} - y = 2t.$$

$\frac{dy}{dt} - \frac{1}{2}y = t$. Let $\mu(t) = e^{-\frac{1}{2}t}$. Multiplying by μ gives

$$e^{-\frac{1}{2}t}\frac{dy}{dt} - \frac{1}{2}e^{-\frac{1}{2}t}y = te^{-\frac{1}{2}t}.$$

So $\frac{d}{dt}\left(e^{-\frac{1}{2}t}y\right) = te^{-\frac{1}{2}t}$

So $e^{-\frac{1}{2}t}y = \int te^{-\frac{1}{2}t}\,dt = -2te^{-\frac{1}{2}t} - \int(-2e^{-\frac{1}{2}t})\,dt$

$= -2te^{-\frac{1}{2}t} - 4e^{-\frac{1}{2}t} + c.$

So $\boxed{y(t) = -2t - 4 + ce^{\frac{1}{2}t}.}$

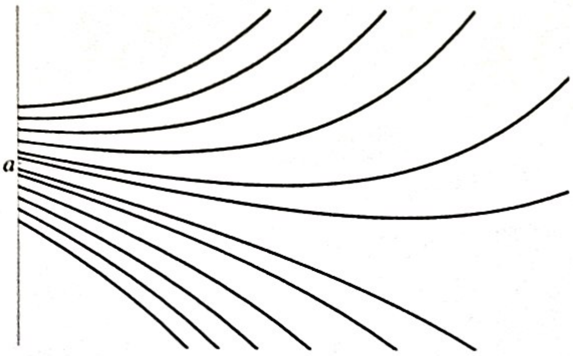
$$y(t) = -2t - 4 + ce^{\frac{1}{2}t}.$$

(c) [5 points] Check your answer to part (b) by calculating $\frac{dy}{dt}$ and $2\frac{dy}{dt} - y$.

$$y' = -2 + \frac{1}{2}ce^{\frac{1}{2}t}.$$

$$2y' - y = \left(\cancel{-4} + \cancel{ce^{\frac{1}{2}t}}\right) - \left(-2t \cancel{-4} + \cancel{ce^{\frac{1}{2}t}}\right) = 2t.$$

Several solutions of $2\frac{dy}{dt} - y = 2t$ are shown below.



(d) [10 points] Now consider the initial value problem

$$\begin{cases} 2\frac{dy}{dt} - y = 2t, \\ y(0) = y_0. \end{cases} \tag{4}$$

As you can see from the graph above, there exists a number $a \in \mathbb{R}$ such that:
- If $y_0 < a$ then $y(t) \to -\infty$ as $t \to \infty$.
- If $y_0 > a$ then $y(t) \to \infty$ as $t \to \infty$.

Find a.

$$y(t) = -2t - 4 + ce^{\frac{1}{2}t}.$$

If $c > 0$, then $y(t) \to \infty$ as $t \to \infty$. If $c < 0$, then $y(t) \to -\infty$ as $t \to \infty$. So we are looking for the solution with $c = 0$. $y(t) = -2t - 4$. Then

$$\boxed{a = y(0) = -4.}$$

(e) [5 points] Describe the behaviour of the solution with initial value $y(0) = a$.

$$y(t) = -2t - 4.$$

$y(t) \to -\infty$ as $t \to \infty$.