OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2013.01.07 **MAT 371 – Diferansiyel Denklemler – Final Sınavın Çözümleri** N. Course

Soru 1 (Intravenous Therapy / Damar İçi Tedavi)**.**

English

A hospital patient is suffering from the Can't-stop-looking-at-his-mobile-phone-in-class disease. To cure this disease, a drug is being administered intravenously to the patient.

Fluid containing 6 mg/cm^3 of the drug enters the patient's blood at a rate of 100 cm^3/hour. The drug is absorbed by the body tissues or otherwise leaves the blood at a rate proportional to the amount of drug in the blood, with a rate constant of $r = 0.3$ (hour)$^{-1}$.

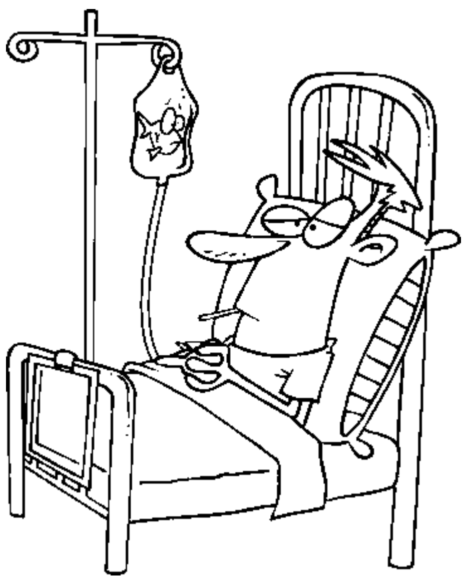
Assume that the drug is always uniformly distributed throughout the patient's blood.

Türkçe

Hastanede yatmakta olan bir hasta "Derste cep telefonuna bakmadan duramama" hastalığına yakalanmıştır. Bu hastalığı tedavi etmek için, hastaya damar yolundan bir ilaç verilmektedir.

6 mg/cm^3 ilaç içeren sıvı, hastanın kan dolaşımına 100 cm^3/saat hızla yayılmaktadır. İlaç, vücut hücreleri tarafından emilmek suretiyle, kanda bulunan ilaç miktarıyla orantılı olarak kanı terketmektedir; hız sabiti $r = 0,3$ (saat)$^{-1}$ dir.

İlacın hastanın kanında daima eşit olarak dağılmış olduğunu varsayın.



(a) [15p] Write a differential equation for the amount of the drug in the patient's blood. (You must explain why your differential equation is valid.)

(b) [10p] How much of the drug is present in the patient's blood after a long time? (i.e. as $t \to \infty$.)

(a) [15p] Hastanın kanındaki ilacın miktarı için bir diferansiyel denklem yazın. (Diferansiyel denkleminizin neden geçerli olduğunu açıklamalısınız.)

(b) [10p] Uzun bir süre sonra hastanın kanında ne kadar ilaç bulunur? ($t \to \infty$)

This is the easiest question on this exam – I hope you chose this one.

(a) Let t denote time measured in hours. Let $b(t)$ denote the amount of the drug, in mg, in the patient's blood at time t. $\boxed{3: \text{we must always say which units we are using!}}$

We would expect the rate of change of b to be

$$\frac{db}{dt} = \begin{pmatrix}\text{amount of drug entering} \\ \text{the blood every hour}\end{pmatrix} - \begin{pmatrix}\text{amount of drug leaving} \\ \text{the blood every hour}\end{pmatrix}. \qquad \boxed{4}$$

Now, the drug enters the blood at a rate of $6\text{mg}/\text{cm}^3 \times 100\text{cm}^3/\text{hour} = 600\text{mg}/\text{hour}$. So

$$\frac{db}{dt} = 600 - \begin{pmatrix}\text{amount of drug leaving} \\ \text{the blood every hour}\end{pmatrix}. \qquad \boxed{4}$$

The question says that the drug leaves the bloodstream at a rate proportional to b with rate constant $0.3/\text{hour}$. In otherwords, the drug leaves at a rate of $0.3/\text{hour} \times b(t)$ mg $= 0.3b$ mg/hour. So

$$\frac{db}{dt} = 600 - 0.3b. \qquad \boxed{4}$$

(b) We don't need to solve the ODE to see this, but it may be easier to understand if we do solve it. Since

$$\begin{aligned}
\frac{db}{dt} &= 600 - 0.3b \\
\frac{db}{b - 2000} &= -0.3dt \\
\log|b - 2000| &= -0.3t + c_1 \\
|b - 2000| &= e^{-0.3t + c_1} \\
b - 2000 &= \pm e^{c_1} e^{-0.3t} \\
b(t) &= 2000 + ce^{-0.3t},
\end{aligned}$$

we can see $\lim_{t\to\infty} b(t) = 2000$. Therefore, after a long time, the amount of drug in the patient's bloodstream will be ≈ 2000 mg.

(Another way to answer this question is to just look for equilibrium solutions to the ODE: $0 = \frac{db}{dt} = 600 - 0.3b \implies b = 2000$. Then you must show that this equilibrium solution is asymptotically stable.)

Soru 2 (Second Order Differential Equations and the Wronskian)**.** Suppose that $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ are continuous functions. Let

$$L[y] = y'' + p(t)y' + q(t)y.$$

(a) [5p] Show that L is a linear operator. In other words: Show that

$$L[u + \lambda v] = L[u] + \lambda L[v]$$

for all twice differentiable functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ and all constants $\lambda \in \mathbb{R}$.

Clearly

$$\begin{aligned}
L[u + \lambda v] &= \big(u(t) + \lambda v(t)\big)'' + p(t)\big(u(t) + \lambda v(t)\big)' + q(t)\big(u(t) + \lambda v(t)\big) \\
&= \big(u''(t) + p(t)u'(t) + q(t)u(t)\big) + \lambda\big(v''(t) + p(t)v'(t) + q(t)v(t)\big) \\
&= L[u] + \lambda L[v].
\end{aligned}$$

(b) [5p] Suppose that

- $y_1(t)$ solves $L[y_1] = 0$;
- $y_2(t)$ solves $L[y_2] = 0$;
- $Y_1(t)$ solves $L[Y_1](t) = g_1(t)$; and
- $Y_2(t)$ solves $L[Y_2](t) = g_2(t)$.
- Define $y(t) = c_1y_1(t) + c_2y_2(t) + Y_1(t) + Y_2(t)$ for $c_1, c_2 \in \mathbb{R}$.

Show that y solves $L[y](t) = g_1(t) + g_2(t)$.

Another easy question: It follows by part (a) that

$$\begin{aligned} L[y](t) &= L\big[c_1y_1 + c_2y_2 + Y_1 + Y_2\big](t) \\ &= c_1L[y_1](t) + c_2L[y_2](t) + L[Y_1](t) + L[Y_2](t) \\ &= 0 + 0 + g_1(t) + g_2(t). \end{aligned}$$

(c) [5p] Calculate the Wronskian of $u(t) = \cos^2 t$ and $v(t) = 1 + \cos 2t$.

$$\begin{aligned} W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos^2 t & 1 + \cos 2t \\ -2\cos t \sin t & -2\sin 2t \end{vmatrix} &= -2\sin 2t \cos^2 t + 2(1 + \cos 2t)\cos t \sin t \\ &= -4\sin t \cos^3 t + 2\cos t \sin t + 2(\cos^2 t - \sin^2 t)\cos t \sin t \\ &= -2\sin t \cos^3 t + 2\cos t \sin t - 2\cos t \sin^3 t \\ &= -2\sin t \cos t(\cos^2 t + \sin^2 t) + 2\cos t \sin t = 0 \end{aligned}$$

(d) [10p] Suppose that $f(t) = t$ and $W(f, g)(t) = t^2e^t$. Find $g(t)$.

Since

$$t^2e^t = W(f, g)(t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = tg' - g$$

we need so solve the first order linear differential equation

$$\frac{dg}{dt} - \frac{1}{t}g = te^t. \qquad \boxed{4}$$

To solve this, we use the integrating factor

$$\mu(t) = e^{\int p(t)dt} = e^{\int -\frac{1}{t}dt} = e^{-\log t} = \frac{1}{t}. \qquad \boxed{3}$$

Multiplying by μ gives

$$\frac{1}{t}\frac{dg}{dt} - \frac{1}{t^2}g = e^t.$$

So

$$\frac{d}{dt}\left(\frac{g}{t}\right) = e^t.$$

Integrating gives

$$\frac{g}{t} = e^t + c.$$

Therefore

$$g(t) = te^t + ct \qquad \boxed{3}$$

for any constant $c \in \mathbb{R}$. [I will also accept the answer $g(t) = te^t$.]

Soru 3 (Second Order Linear Differential Equations)**.** [25p] Solve

$$\begin{cases} -y'' + 6y' - 16y = 1 + 6e^{3t}\sin(2t) \\ y(0) = \frac{15}{16} \\ y'(0) = -1 \end{cases} \tag{1}$$

First consider the homogeneous equation $-y'' + 6y' - 16y = 0$. The characteristic equation is $-r^2 + 6r - 16 = 0$ which has roots $r = 3 \pm i\sqrt{7}$. Therefore the general solution of $-y'' + 6y' - 16y = 0$ is

$$y(t) = c_1 e^{3t}\sin(\sqrt{7}t) + c_2 e^{3t}\cos(\sqrt{7}t). \qquad \boxed{4}$$

Next consider $-y'' + 6y' - 16y = 1$. Trying the ansatz $Y(t) = C$, we see that

$$1 = -Y'' + 6Y' - 16Y = -16C.$$

We must choose $C = -\frac{1}{16}$. Hence $Y(t) = -\frac{1}{16}$. $\boxed{4}$

Now consider $-y''+6y'-16y = 6e^{3t}\sin(2t)$. We try the ansatz $Y(t) = Ae^{3t}\cos 2t + Be^{3t}\sin 2t$ and find that

$$\begin{aligned} 6e^{3t}\sin 2t &= -Y'' + 6Y' - 16Y \\ &= -e^{3t}\Big((5A + 12B)\cos 2t + (5B - 12A)\sin 2t\Big) \\ &\quad + 6e^{3t}\big((3A + 2B)\cos 2t + (3B - 2A)\sin 2t\big) \\ &\quad - 16e^{3t}\big(A\cos 2t + B\sin 2t\big) \\ &= e^{3t}\cos 2t\big(-5A - 12B + 16A + 12B - 16A\big) \\ &\quad + e^{3t}\sin 2t\big(-5B + 12A + 18B - 12A - 16B\big) \\ &= e^{3t}\cos 2t\big(-5A\big) + e^{3t}\sin 2t\big(-3B\big) \end{aligned}$$

Thus, we need $A = 0$ and $B = -2$. Hence

$$Y(t) = -2e^{3t}\sin 2t \qquad \boxed{8}$$

Next we add these 3 solutions together. Therefore, the general solution of the ODE is

$$y(t) = c_1 e^{3t}\sin(\sqrt{7}t) + c_2 e^{3t}\cos(\sqrt{7}t) - 2e^{3t}\sin(2t) - \frac{1}{16}. \qquad \boxed{4}$$

The final step is to satisfy the initial conditions.

$$\frac{15}{16} = y(0) = 0 + c_2 - 0 - \frac{1}{16} \qquad \Longrightarrow \qquad c_2 = 1.$$

$$\begin{aligned} -1 &= y'(0) \\ &= 3c_1 e^{3t}\sin(\sqrt{7}t) + \sqrt{7}c_1 e^{3t}\cos(\sqrt{7}t) + 3e^{3t}\cos(\sqrt{7}t) - \sqrt{7}e^{3t}\sin(\sqrt{7}t) \\ &\quad -6e^{3t}\sin(2t) - 4e^{3t}\cos(2t)\big|_{t=0} \\ &= 0 + \sqrt{7}c_1 + 3 - 0 - 0 - 4 \qquad \Longrightarrow \qquad c_1 = 0. \end{aligned}$$

Therefore, the solution of (1) is

$$y(t) = e^{3t}\cos(\sqrt{7}t) - 2e^{3t}\sin(2t) - \frac{1}{16}. \qquad \boxed{5}$$

Soru 4 (Systems of Equations)**.**

(a) [11p] Solve

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ are $r_1 = -3$ and $r_2 = -1$. $\boxed{2}$ The eigenvectors are $\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\boxed{2}$.

Therefore, the general solution of $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \boxed{4}.$$

Finally, we use the initial condition to calculate that

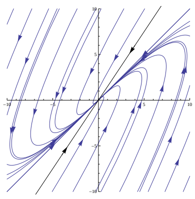
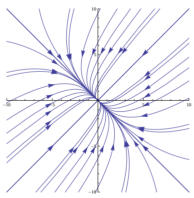
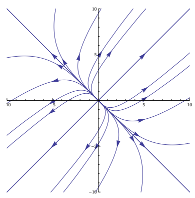
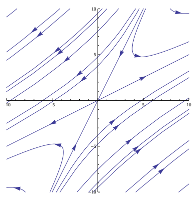
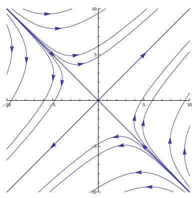
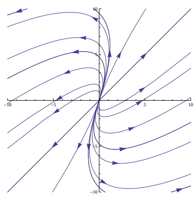
$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_2 - c_1 \end{pmatrix}$$

which tells us that $c_1 = -\frac{1}{2}$ and $c_2 = \frac{7}{2}$. Thus, the answer to this question is

$$\mathbf{x}(t) = -\frac{1}{2} e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{7}{2} e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \boxed{3}.$$

(b) [2p] How does the solution behave as $t \to \infty$?

$\mathbf{x}(t) \to 0$ as $t \to \infty$. More precisely, the solution approaches the origin along the line $x_2 = x_1$ as $t \to \infty$.

(i) Stable node (ii) Stable node (iii) Unstable node (iv) Saddle (v) Saddle (vi) Unstable node

Let $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$. The determinant of A is 2 and the trace of A is -3. The eigenvalues of A are $r_1 = -2$ and $r_2 = -1$. The corresponding eigenvectors of A are $\xi^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

(c) [2p] Which of the graphs (above) is the phase plot of the equation $\mathbf{x}' = A\mathbf{x}$?

[Mark ☑ one box only.]

☑ (i) ☐ (ii) ☐ (iii) ☐ (iv) ☐ (v) ☐ (vi)

(d) [10p] Justify (explain) your answer to part (c).

Since both r_1 and r_2 are strictly negative (< 0), we must have a stable node. So the phase plot must be either (i) or (ii). $\boxed{5}$

The phase plot must also have straight lines in the directions of the eigenvectors. So it must be (i). $\boxed{5}$

Soru 5 (Exact Equations)**.** Consider

$$\left(2x^2e^{x^2}\log|y|\right) + \left(\frac{xe^{x^2}}{y} - x\sinh y\right)\frac{dy}{dx} = 0 \tag{2}$$

This equation is of the form $M(x,y) + N(x,y)y' = 0$.

(a) [2p] Is this equation exact?

No $\boxed{2}$, because

$$M_y = \frac{2x^2e^{x^2}}{y} \neq \frac{e^{x^2} + 2x^2e^{x^2}}{y} - \sinh y = N_x. \qquad \boxed{2}$$

(b) [2p] Calculate $\frac{M_y - N_x}{N}$ and $\frac{N_x - M_y}{M}$.

$$\frac{M_y - N_x}{N} =$$

$$\frac{N_x - M_y}{M} =$$

$$\frac{M_y - N_x}{N} = \frac{\sinh y - \frac{e^{x^2}}{y}}{\frac{xe^{x^2}}{y} - x\sinh y} = -\frac{1}{x}$$

$$\frac{N_x - M_y}{M} = \frac{\frac{e^{x^2} - \sinh y}{y}}{2x^2e^{x^2}\log|y|}$$

(c) [6p] Find an integrating factor $\mu(x)$ that solves

$$\frac{d\mu}{dx}(x) = \mu(x)\cdot\left(\frac{M_y - N_x}{N}\right)$$

$$\begin{aligned}\frac{d\mu}{dx} &= -\frac{\mu}{x}\\ \frac{d\mu}{\mu} &= -\frac{dx}{x}\\ \log|\mu| &= -\log|x| + C\\ \mu(x) &= \frac{c}{x}.\end{aligned}$$

We can choose $c = 1$ to get $\mu(x) = \frac{1}{x}$.

$$\left(2x^2e^{x^2}\log|y|\right) + \left(\frac{xe^{x^2}}{y} - x\sinh y\right)\frac{dy}{dx} = 0 \tag{2}$$

(d) [1p] Multiply (2) by the integrating factor that you found in part (c).

$$\left(2xe^{x^2}\log|y|\right) + \left(\frac{e^{x^2}}{y} - \sinh y\right)\frac{dy}{dx} = 0 \tag{3}$$

(e) [2p] Show that (3) is exact?

[HINT: Equation (3) is your answer to part (d). If (3) is not exact, then your answer to part (c) is probably wrong.]

$$M_y = \frac{2xe^{x^2}}{y}$$
$$N_x = \frac{2xe^{x^2}}{y} = M_y$$

Therefore (3) is exact.

(f) [12p] Solve (3).

We need to find a function $\psi(x, y)$ such that

$$\psi_x = 2xe^{x^2} \log |y|$$
$$\psi_y = \frac{e^{x^2}}{y} - \sinh y.$$

Integrating the first equation wrt x gives

$$\psi(x, y) = e^{x^2} \log |y| + h(y)$$

for some function $h(y)$. Then we differentiate wrt y to see that

$$\psi_y = \frac{e^{x^2}}{y} + h'(y).$$

We need to choose $h(y) = -\cosh y$ to satisfy the ψ_y equation above. Therefore, the solution to (3) is

$$e^{x^2} \log |y| - \cosh y = c$$

for some constant c.

Note, there was a misprint on the exam regarding points per question part for question 5.

I was generous when marking to make up for this.