2013.11.14 **MAT 371 – Diferansiyel Denklemler – Ara Sınavın Çözümleri** N. Course

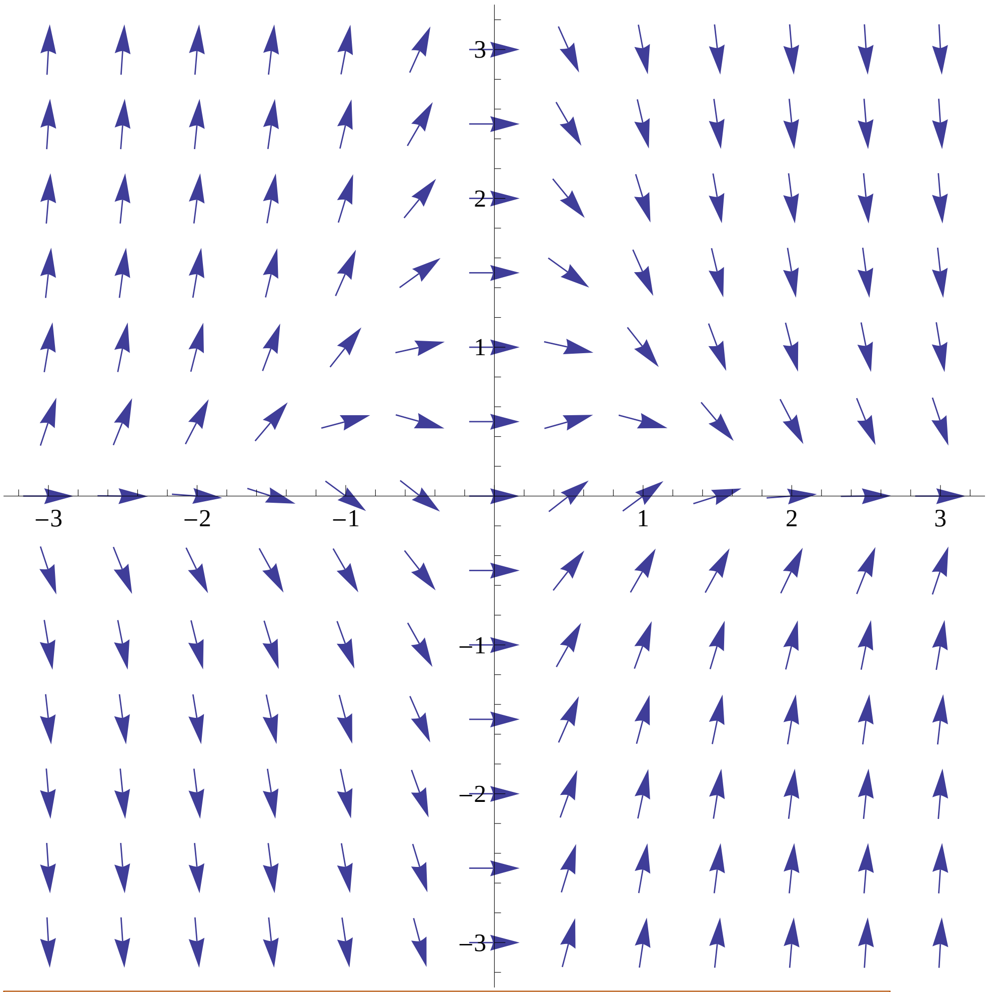
Soru 1 (Linear Equations).

(a) [25p] Draw a direction field for

$$\frac{dy}{dt} = 2t\left(e^{-t^2} - y\right).$$

[HINT: I want to see an arrow on every dot, and on every mark on the axes.]

[HINT: $e^{-1} \approx 0.37$, $e^{-4} \approx 0.02$, $e^{-9} \approx 0.0001$.]



-0.1 point for each arrow missing/incorrect, then rounded to nearest point.

(b) [25p] Solve

$$\begin{cases} \frac{dy}{dt} = 2t\left(e^{-t^2} - y\right) \\ y(5) = 0. \end{cases} \tag{1}$$

First we rewrite the ODE as

$$\frac{dy}{dt} + 2ty = 2te^{-t^2}.$$

This is a linear ODE ($y' + p(t)y = g(t)$) with $p(t) = 2t$ $\boxed{5}$. We use the integrating factor

$$\mu(t) = e^{\int p(t)dt} = e^{\int 2t\ dt} = e^{t^2}. \ \boxed{5}$$

Multiplying the ODE by μ, we obtain

$$\frac{d}{dt}\left[e^{t^2}y\right] = e^{t^2}\frac{dy}{dt} + 2te^{t^2} = 2t. \ \boxed{5}$$

Then we can integrate to see

$$e^{t^2}y(t) = t^2 + c.$$

Thus

$$y(t) = t^2e^{-t^2} + ce^{-t^2}. \ \boxed{5}$$

Finally, we use the initial condition $y(5) = 0$ to see that $c = -25$. Therefore

$$y = (t^2 - 25)e^{-t^2}. \ \boxed{5}$$

Soru 2 (Autonomous Equations)**.** Consider the autonomous differential equation

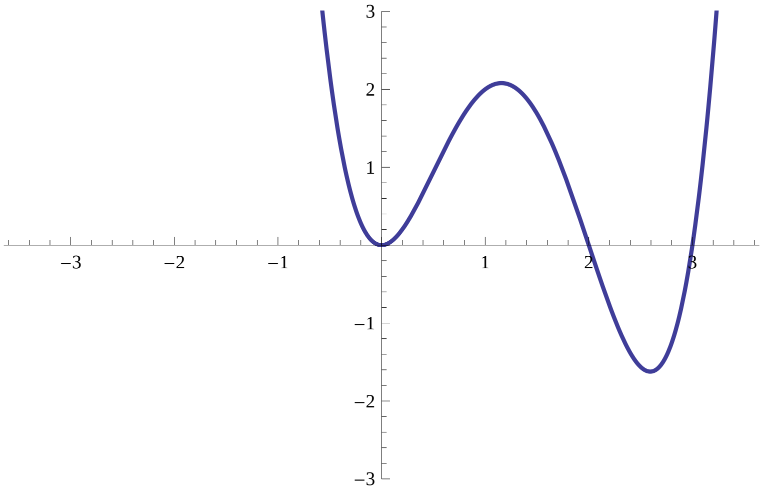
$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2. \tag{2}$$

(a) [10p] Find all of the critical points of (2).

$$\frac{dy}{dt} = f(y) = y^4 - 5y^3 + 6y^2 = y^2(y-2)(y-3).$$

The critical points are $y = 0$, $y = 2$ and $y = 3$.

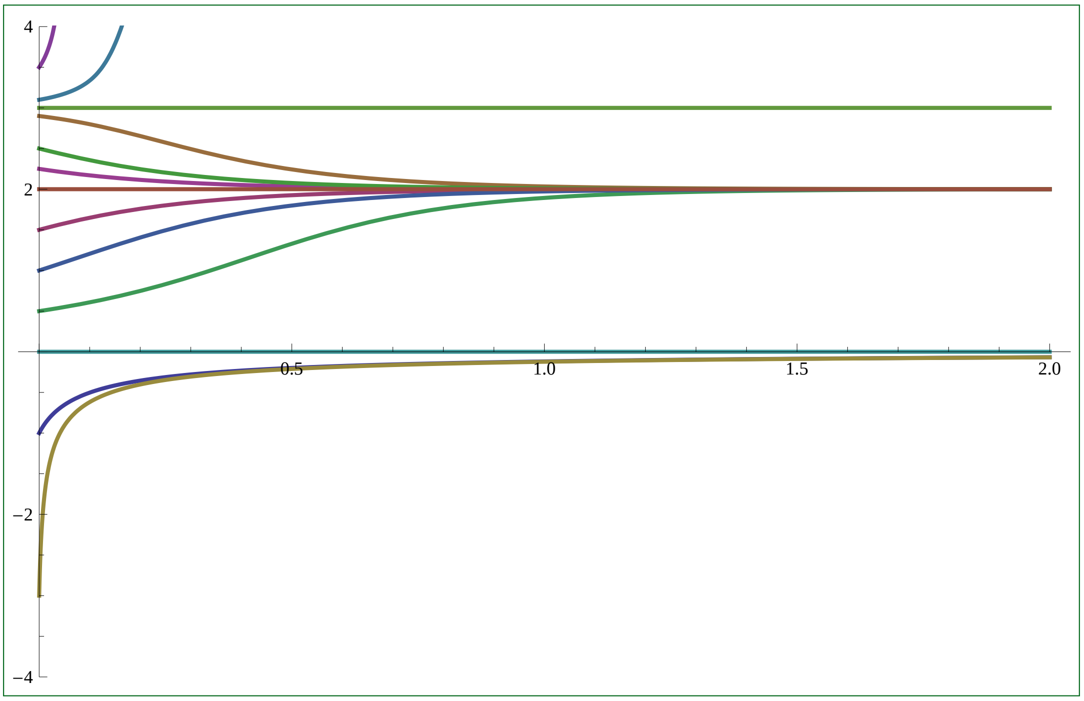
(b) [15p] Sketch the graph of $f(y)$ versus y.



(c) [9p] Determine whether each critical point is asymptotically stable, unstable or semistable.

$y = 0$ is semistable, $y = 2$ is asymptotically stable and $y = 3$ is unstable.

(d) [16p] Sketch 10 (or more) different solutions of (2).



Soru 3 (Separable Equations)**.** Consider the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}. \tag{3}$$

(a) [5p] Show that this differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}. \tag{4}$$

This is easy:

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy} == \frac{1 + \frac{3y^2}{x^2}}{\frac{2xy}{x^2}} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

Let $v(x)$ be a new variable such that $v = y/x$ (or $y(x) = xv(x)$).

(b) [15p] Use (4) to show that

$$v + x\frac{dv}{dx} = \frac{1 + 3v^2}{2v}$$

and

$$x\frac{dv}{dx} = \frac{1 + v^2}{2v}.$$

$$\frac{1 + 3v^2}{2v} = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} = \frac{dy}{dx} = \frac{d}{dx}(xv) = v + x\frac{dv}{dx} \quad \boxed{10}$$

and

$$x\frac{dv}{dx} = \frac{1 + 3v^2}{2v} - v = \frac{1 + 3v^2}{2v} - \frac{2v^2}{2v} = \frac{1 + v^2}{2v}. \quad \boxed{5}$$

(c) [20p] The equation

$$x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

is a separable differential equation. Solve this equation.

[HINT: $\int \frac{2t}{1+t^2}dt = \log(1+t^2) + \text{constant}$.]

First we separate the variables

$$\frac{2v}{1+v^2}dv = \frac{dx}{x}$$

then we integrate

$$\begin{aligned}\int \frac{2v}{1+v^2}dv &= \int \frac{dx}{x}\\ \log(1+v^2) &= \log|x| + c\\ 1+v^2 &= Cx\end{aligned}$$

(d) [10p] Find an explicit solution to

$$\begin{cases}\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}\\ y(1) = 2.\end{cases}$$

[HINT: Remember $v = y/x$.]

Since $v = \frac{y}{x}$ and $1 + v^2 - Cx = 0$ we have that

$$x^2 + y^2 - Cx^3 = 0.$$

Since $y(1) = 2$, we must have that $C = 5$ $\boxed{5}$. Rearranging we see that

$$y(x) = \pm\sqrt{5x^3 - x^2}.$$

Finally we use the initial condition again to see that we must have "+". So

$$y(x) = \sqrt{5x^3 - x^2}. \boxed{5}$$