2015.01.08 **MAT371 Diferansiyel Denklemler – Final Sınavın Çözümleri** N. Course

Soru 1 (Petrol Tank / Benzin Deposu).

English

In an oil refinery, a storage tank contains 8000 litres of petrol[1] that initially contains 50kg of an additive dissolved in it.

In preparation for winter weather, petrol containing 0.25 kg/litre of the additive is pumped into the tank at a rate of 160 litres/minute.

The well-mixed solution is pumped out at a rate of 180 litres/minute.

Türkçe

Bir petrol rafinerisindeki bir depoda, başlangıçta içinde 50kg katkı maddesi çözeltilmiş olan 8000 litre benzin bulunmaktadır.

Kış şartlarına hazırlık olarak, içinde 0.25kg/litre katkı maddesi olan benzin 160 litre/dakika hızla depoya pompalanmaktadır. İyice karışmış olan karışım 180 litre/dakika hızla depodan dışarı pompalanmaktadır.

(a) [15p] Write an initial value problem (IVP) for the amount of additive in the tank at time t.
(You must explain why your differential equation is valid.)

(b) [9p] Solve the IVP that you wrote in part (a).
[HINT: Use the integrating factor $\mu(t) = e^{\int p(t)dt}$.]

(c) [1p] How much additive is in the tank after 400 minutes?

(a) [15p] Depodaki katkı maddesi miktarı için (t zamanda) bir başlangıç değer problemi (IVP) yazınız.
(Diferansiyel denkleminizin neden geçerli olduğunu açıklamalısınız.)

(b) [9p] (a) bölümünde yazmış olduğunuz IVP'yi çözünüz.

(c) [1p] 400 dakika sonra depoda ne kadar katkı maddesi bulunmaktadır?

[1] or "gasoline" in American English

(a) Let t denote time measured in minutes. Let $a(t)$ denote the amount of additive, in kg, in the tank at time t. $\boxed{3: \text{we must always say which units we are using!}}$ We know that $a(0) = 50$ kg.

Clearly, we have 0.25 kg/litre ×160 litres/minute = 40 kg/minute of additive entering the tank.

Until the tank is empty, the volume of petrol in the tank at time t is

$$\begin{aligned} V(t) &= 8000 \text{ litres } + \Big(160 \text{ litres/minute } - 180 \text{ litres/minute}\Big) \times t \text{ minutes} \\ &= (8000 - 20t) \text{ litres.} \end{aligned}$$

Therefore, we have $\frac{a(t)\text{ kg}}{V(t)\text{ litres}} \times 180$ litres/minute $= \frac{180a(t)}{8000-20t}$ kg/minute of additive leaving the tank.

Hence, our IVP is

$$\begin{cases} \frac{da}{dt} + \frac{9}{400-t}a(t) = 40, \\ a(0) = 50. \end{cases}$$

This IVP is valid for $0 \le t \le 400$.

(b) Our ODE is a linear first order equation, so we use the integrating factor

$$\begin{aligned} \mu(t) &= e^{\int p(t)dt} = e^{\int \frac{9}{400-t}dt} \\ &= e^{-9\log(400-t)} = (400-t)^{-9}. \end{aligned}$$

So

$$\begin{aligned} \frac{d}{dt}\left((400-t)^{-9}a\right) &= (400-t)^{-9}\frac{da}{dt} + 9(400-t)^{-10}a(t) \\ &= 40(400-t)^{-9}. \end{aligned}$$

Integrating gives

$$\begin{aligned} (400-t)^{-9}a &= 40\int (400-t)^{-9}dt \\ &= 40\frac{(400-t)^{-8}}{-1\times -8} + C \\ &= 5(400-t)^{-8} + C. \end{aligned}$$

Therefore

$$a(t) = 5(400-t) + C(400-t)^9$$

(c) This part is easy; after 400 minutes the tank will be empty. So the answer is: There will be 0 kg of additive in the tank after 400 minutes.

Soru 2 (Second Order Linear Differential Equations). [25p] Solve

$$25y'' - 20y' + 4y = t + 841\cos t - 7e^{2t}. \tag{1}$$

First consider the homogeneous equation $25y'' - 20y' + 4y = 0$. The characteristic equation is $25r^2 - 20r + 4 = 0$ which has repeated root $r = \frac{2}{5}$. Therefore the general solution of $25y'' - 20y' + 4y = 0$ is

$$y(t) = c_1 e^{\frac{2t}{5}} + c_2 t e^{\frac{2t}{5}}.$$

Next consider $25r^2 - 20r + 4 = t$. Trying the ansatz $Y(t) = At + B$, we see that

$$t = 25Y'' - 20Y' + 4Y = 25(0) - 20(A) + 4(At + B) = 4At + (-20A + 4B).$$

We must choose $A = \frac{1}{4}$ and $B = \frac{5}{4}$. Hence

$$Y(t) = \frac{t}{4} + \frac{5}{4}.$$

Now consider $25y'' - 20y' + 4y = 841\cos t$. We try the ansatz $Y(t) = A\cos t + B\sin t$ and find that

$$\begin{aligned} 841\cos t &= 25Y'' - 20Y' + 4Y \\ &= 25\Big(- A\cos t - B\sin t\Big) - 20\Big(B\cos t - A\sin t\Big) + 4\Big(A\cos t + B\sin t\Big) \\ &= \Big[- 21A - 20B\Big]\cos t + \Big[- 21B + 20A\Big]\sin t \end{aligned}$$

Thus, we need $A = -21$ and $B = -20$. Hence

$$Y(t) = -21\cos t - 20\sin t$$

Next consider $25y'' - 20y' + 4y = -7e^{2t}$. We try the ansatz $Y(t) = Ae^{2t}$ and we find that

$$\begin{aligned} -7e^{2t} &= 25Y'' - 20Y' + 4Y \\ &= 25\Big(4Ae^{2t}\Big) - 20\Big(2Ae^{2t}\Big) + 4\Big(Ae^{2t}\Big) \\ &= 64Ae^{2t}. \end{aligned}$$

So, we need to choose $A = -\frac{7}{64}$. Hence

$$Y(t) = -\frac{7}{64}e^{2t}.$$

Finally we add these 4 solutions together. Therefore, the general solution to the ODE is

$$y(t) = c_1 e^{\frac{2t}{5}} + c_2 t e^{\frac{2t}{5}} + \frac{t}{4} + \frac{5}{4} - 21\cos t - 20\sin t - \frac{7}{64}e^{2t}.$$

Soru 3 (Reduction of Order)**.** Consider

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0, \qquad x > 0. \tag{2}$$

(a) [2p] Show that $y_1(x) = x^{-1}$ is a solution of (2).

Clearly

$$x^2y_1'' + 3xy_1' + y_1 = x^2(2x^{-3}) + 3x(-x^{-2}) + (x^{-1}) = 2x^{-1} - 3x^{-1} + x^{-1} = 0.$$

(b) [17p] Using the method of reduction of order, find a second solution $y_2(x)$ of (2).
[HINT: Start with $y_2(x) = v(x)y_1(x)$.]

As per the hint, we start with $y_2(x) = v(x)x^{-1}$. Then $y_2' = v'x^{-1} - vx^{-2}$ and $y_2'' = v''x^{-1} - 2v'x^{-2} + 2vx^{-3}$. Putting these into the differential equation, we get

$$\begin{aligned} 0 &= x^2y_2'' + 3xy_2' + y_2 \\ &= (xv'' - 2v' + 2vx^{-1}) + (3v' - 3vx^{-1}) + vx^{-1} \\ &= xv'' + v' \\ &= xu' + u \end{aligned}$$

where $u = v'$. We must solve $x\frac{du}{dx} + u = 0$. Rearranging gives $\frac{du}{u} = -\frac{dx}{x}$. Integrating gives $\log|u| = -\log|x| + c$. Rearranging then gives $u = Cx^{-1}$ for some constant C.

Next we integrate to find $v(x) = \int u(x)dx = C\log x + c$. For simplicity, choose $c = 0$ and $C = 1$ to get $v(x) = \log x$. Then $y_2(x) = v(x)x^{-1} = x^{-1}\log x$.

(c) [2p] Check that **your** $y_2(x)$ is a solution of (2).

If $y_2(x) = x^{-1}\log x$, then

$$\begin{aligned} x^2y_2'' + 3xy_2' + y_2 &= x^2(-2x^{-3} + 2x^{-3}\log x - x^{-3}) + 3x(x^{-2} - x^{-2}\log x) + (x^{-1}\log x) \\ &= -2x^{-1} + 2x^{-1}\log x - x^{-1} + 3x^{-1} - 3x^{-1}\log x + x^{-1}\log x \\ &= 0. \end{aligned}$$

(d) [4p] Show that $y_1(x) = x^{-1}$ and **your** y_2 are linearly independent.

We calculate the Wronskian

$$W(y_1, y_2)(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} x^{-1} & x^{-1}\log x \\ -x^{-2} & x^{-2} - x^{-2}\log x \end{pmatrix} = x^{-3} \neq 0.$$

Therefore y_1 and y_2 are linearly independent.

Soru 4 (Systems of Equations)**.**

(a) [13p] Solve

$$\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}$ are $r_1 = \frac{1}{2}$ and $r_2 = 2$. The eigenvectors are $\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Therefore, the general solution of $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}$ is

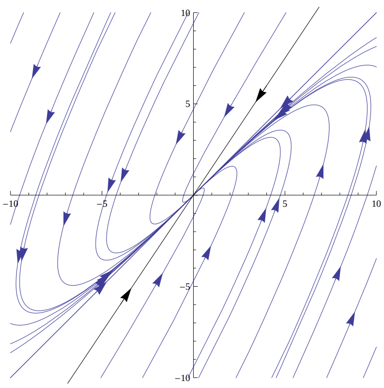
$$\mathbf{x}(t) = c_1 e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Finally, we use the initial condition to calculate that

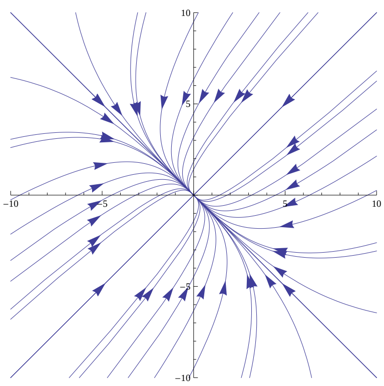
$$\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_2 - c_1 \end{pmatrix}$$

which tells us that $c_1 = \frac{1}{2}$ and $c_2 = \frac{9}{2}$. Thus, the answer to this question is

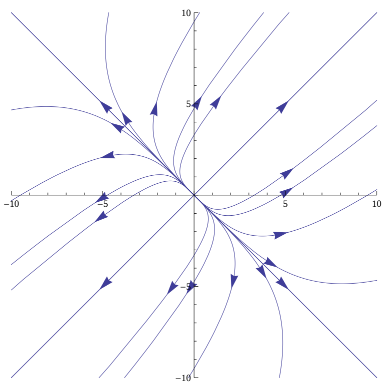
$$\mathbf{x}(t) = \frac{1}{2} e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{9}{2} e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



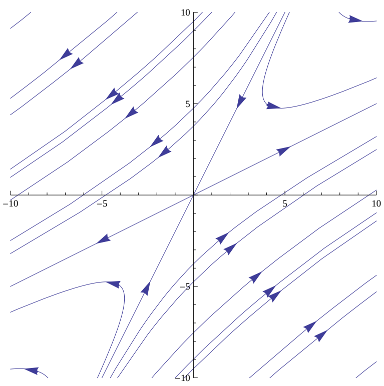
(i) Stable node



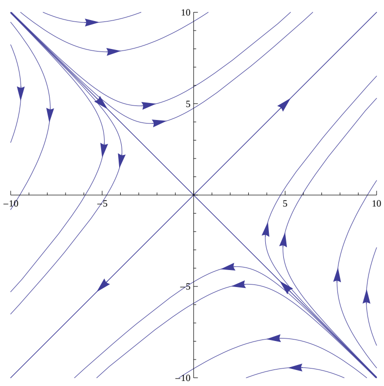
(ii) Stable node



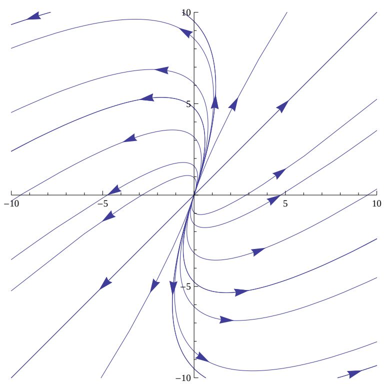
(iii) Unstable node



(iv) Saddle



(v) Saddle



(vi) Unstable node

Let $A = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}$. The determinant of A is -16 and the trace of A is 6. The eigenvalues of A are $r_1 = 8$ and $r_2 = -2$. The corresponding eigenvectors of A are $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

(b) [2p] Which of the graphs (above) is the phase plot of the equation $\mathbf{x}' = A\mathbf{x}$?

[Mark ☑ one box only.]

☐ (i) ☐ (ii) ☐ (iii) ☐ (iv) ☑ (v) ☐ (vi)

(c) [10p] Justify (explain) your answer to part (b).

Since one eigenvalue is positive and one is negative, we must have a saddle point. So the phase plot must be either (iv) or (v).

The phase plot must also have straight lines in the directions of the eigenvectors. So it must be (v).

Soru 5 (Bernoulli Differential Equation). Consider the differential equation

$$\frac{dy}{dt} - y = -y^2. \tag{3}$$

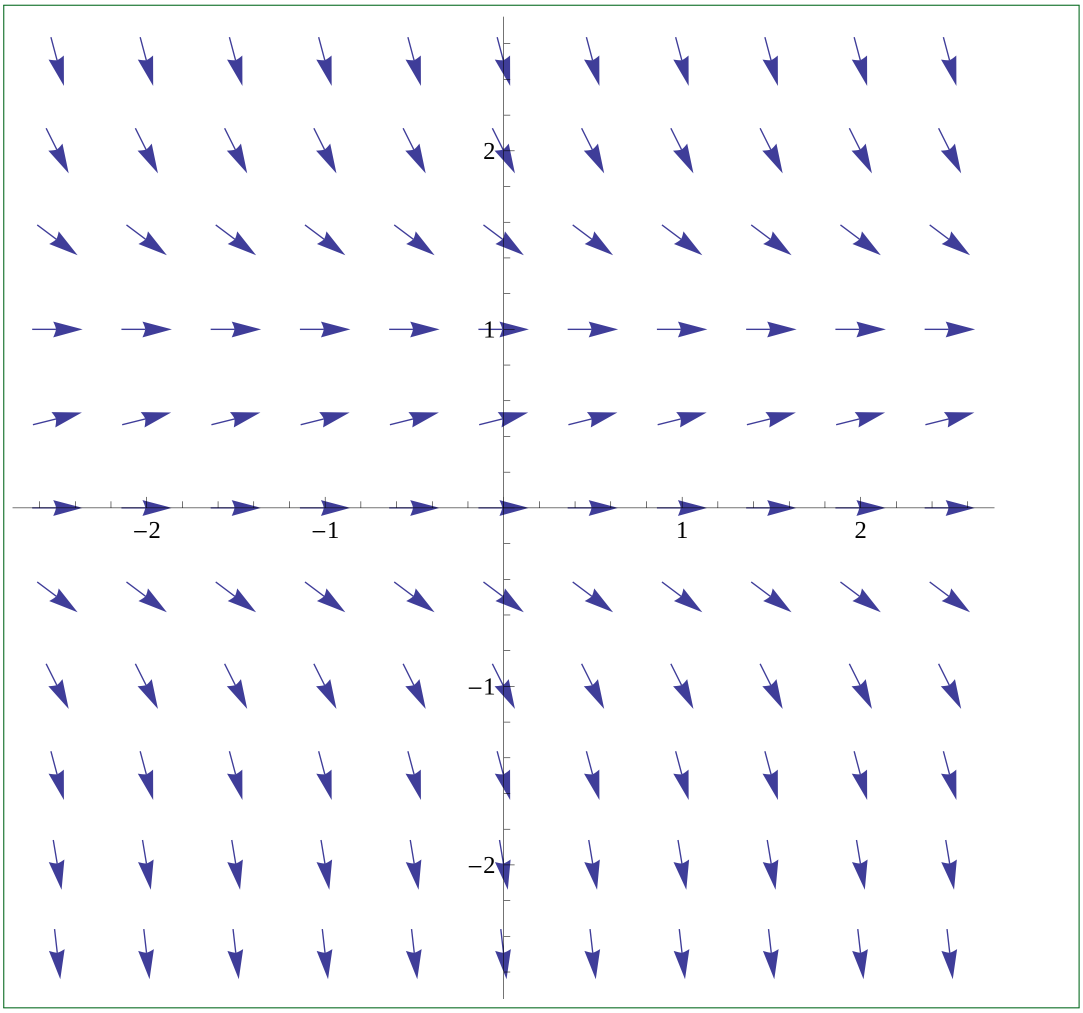
(a) [1p] What is the order of (3)?

☐ 5th order ☐ 2nd order ☐ 3rd order ☑ 1st order ☐ 99th order

(b) [1p] Is (3) linear or non-linear?

☐ linear ☑ non-linear

(c) [10p] Draw a direction field for (3).



$$\frac{dy}{dt} - y = -y^2 \tag{3}$$

Define $u(t) = \frac{1}{y(t)}$.

(d) [5p] Show that

$$\frac{du}{dt} + u = 1 \tag{4}$$

Since

$$\frac{du}{dt} = \frac{d}{dt}\left(\frac{1}{y}\right) = \frac{dy}{dt}\frac{d}{dy}\left(\frac{1}{y}\right) = \frac{dy}{dt}\Big(- y^{-2}\Big)$$

we have that

$$u' + u = -y^{-2}y' + y^{-1} = -y^{-2}\left(y' - y\right) = -y^{-2}\left(-y^2\right) = 1$$

(e) [5p] Solve (4).

A simple linear equation: We use the integrating factor $\mu(t) = e^t$ to see that

$$\left(e^t u\right)' = e^t u' + e^t u = e^t.$$

Integrating and rearranging, we have that

$$e^t u = e^t + C$$
$$u(t) = 1 + Ce^{-t}.$$

(f) [3p] Now solve

$$\begin{cases} \frac{dy}{dt} - y = -y^2, \\ y(0) = 2. \end{cases}$$

The general solution to $y' - y = -y^2$ is

$$y(t) = \frac{1}{u(t)} = \frac{1}{1 + Ce^{-t}}.$$

Using the initial condition, we find that $2 = \frac{1}{1+C}$ which implies that $C = -\frac{1}{2}$. Therefore

$$y(t) = \frac{1}{1 - \frac{1}{2}e^{-t}} = \frac{2}{2 - e^{-t}}.$$