2014.11.20 **MAT371 Diferansiyel Denklemler – Ara Sınavın Çözümleri** N. Course

Soru 1 (Exact Equations). Consider

$$\left(2xye^y + 2ye^{-x^2}\right) + \left(e^y + ye^y + 2xe^{-x^2}\right)\frac{dy}{dx} = 0 \tag{1}$$

This equation is of the form $M(x,y) + N(x,y)y' = 0$.

(a) [4p] Is this equation exact?

Since $M_y = 2xe^y + 2xye^y + 2e^{-x^2}$ and $N_x = 2e^{-x^2} - 4x^2e^{-x^2}$, (1) is not exact.

(b) [4p] Show that $\frac{M_y - N_x}{N} = 2x$.

$$\begin{aligned}\frac{M_y - N_x}{N} &= \frac{\left(2xe^y + 2xye^y + 2e^{-x^2}\right) - \left(2e^{-x^2} - 4x^2e^{-x^2}\right)}{e^y + ye^y + 2xe^{-x^2}} \\ &= \frac{2xe^y + 2xye^y + 4x^2e^{-x^2}}{e^y + ye^y + 2xe^{-x^2}} \\ &= \frac{2x\left(e^y + ye^y + 2xe^{-x^2}\right)}{e^y + ye^y + 2xe^{-x^2}} = 2x.\end{aligned}$$

(c) [12p] Find an integrating factor $\mu(x)$ that solves $\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$.

We must solve

$$\frac{d\mu}{dx} = 2x\mu.$$

Rearranging, integrating and rearranging gives

$$\begin{aligned}\frac{d\mu}{\mu} &= 2x\ dx \\ \int \frac{d\mu}{\mu} &= \int 2x\ dx \\ \log|\mu| &= x^2 + C \\ |\mu| &= e^C e^{x^2} \\ \mu &= \pm e^C e^{x^2} = ce^{x^2}.\end{aligned}$$

For simplicity, I choose $c = 1$. My integrating factor is $\mu(x) = e^{x^2}$.

(d) [1p] Multiply (1) by the integrating factor that you found in part (c).
[This new equation will be called (2).]

$$\left(2xye^{x^2+y} + 2y\right) + \left(e^{x^2+y} + ye^{x^2+y} + 2x\right)\frac{dy}{dx} = 0 \tag{2}$$

(e) [4p] Show that (2) is exact?

[HINT: If (2) is not exact, then your answer to part (c) is probably wrong.]

Now $M = 2xye^{x^2+y} + 2y$, $M_y = 2xe^{x^2+y} + 2xye^{x^2+y} + 2$, $N = e^{x^2+y} + ye^{x^2+y} + 2x$ and $N_x = 2xe^{x^2+y} + 2xye^{x^2+y} + 2$. Since $M_y = N_x$, (2) is exact.

(f) [25p] Solve (2).

We must find a function $\psi(x, y)$ such that

$$\psi_x(x, y) = M = 2xye^{x^2+y} + 2y \qquad \text{and} \qquad \psi_y(x, y) = N = e^{x^2+y} + ye^{x^2+y} + 2x.$$

Integrating the former equation (wrt x) gives

$$\psi = \int \psi_x \, dx + h(y) = \int 2xye^{x^2+y} + 2y \, dx + h(y) = ye^{x^2+y} + 2xy + h(y).$$

Then differentiating (wrt y) gives

$$\psi_y = e^{x^2+y} + ye^{x^2+y} + 2x + h'(y)$$

Since we need $h'(y) = 0$ for all y, we can choose $h(y) \equiv 0$. Therefore

$$\psi = ye^{x^2+y} + 2xy.$$

Hence the solution to (2) (and also the solution to (1)) is

$$ye^{x^2+y} + 2xy = c$$

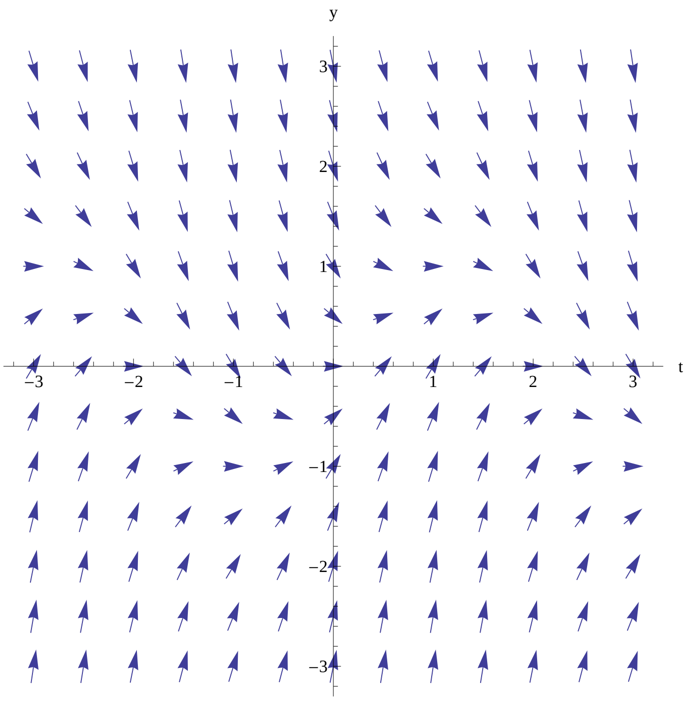
for a constant c.

Soru 2 (a).

(a) [25p] Draw a direction field for the equation

$$\frac{dy}{dt} + y = \sin\frac{\pi t}{2}. \tag{3}$$

[HINT: I want to see 225 arrows – one on every dot, and one on every mark on the axes.]



(b) [20p] Find the general solution to

$$\frac{dy}{dt} + y = \sin\frac{\pi t}{2}. \tag{3}$$

We use the integrating factor $\mu(t) = e^t$ $\boxed{4}$. Therefore

$$\frac{d}{dt}(e^t y) = e^t y' + e^t y = e^t \sin\frac{\pi t}{2}.\boxed{3}$$

Integrating gives

$$e^t y = \int e^t \sin\frac{\pi t}{2} dt\boxed{3} = -\frac{2e^t(\pi\cos\frac{\pi t}{2} - 2\sin\frac{\pi t}{2})}{4+\pi^2} + c.\boxed{9}$$

Finally, we see that the general solution to (3) is

$$y(t) = \frac{4\sin\frac{\pi t}{2} - 2\pi\cos\frac{\pi t}{2}}{4+\pi^2} + ce^{-t}.\boxed{1}$$

(c) [5p] Solve

$$\begin{cases} \frac{dy}{dt} + y = \sin\frac{\pi t}{2} \\ y(0) = 3. \end{cases}$$

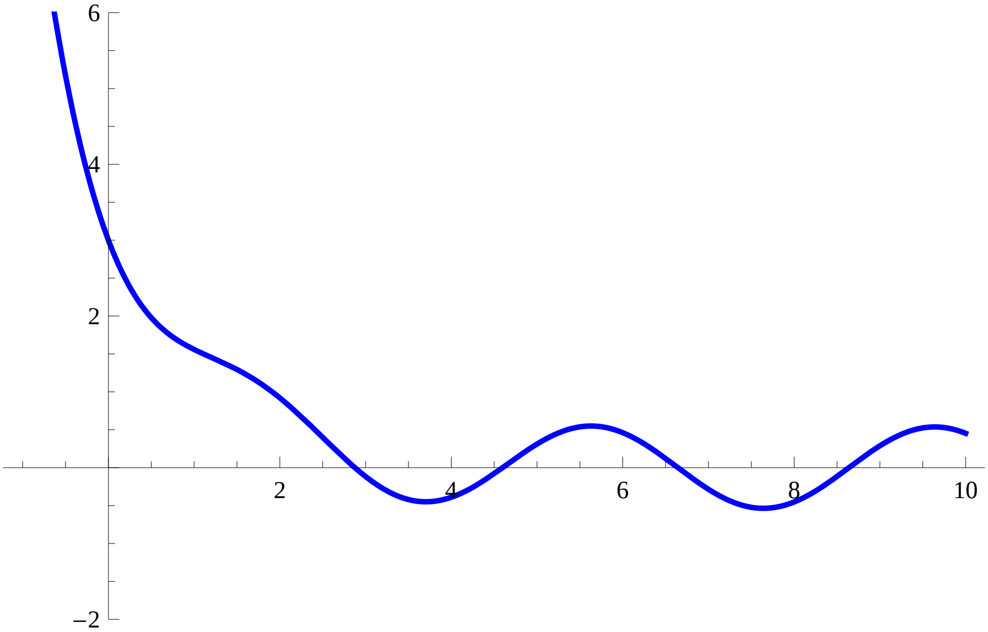
Using the initial condition, we can see that

$$3 = y(0) = \frac{4\sin 0 - 2\pi\cos 0}{4+\pi^2} + ce^0 = -\frac{2\pi}{4+\pi^2} + c \qquad \Longrightarrow \qquad c = 3 + \frac{2\pi}{4+\pi^2}.$$

Therefore

$$y(t) = \frac{4\sin\frac{\pi t}{2} - 2\pi\cos\frac{\pi t}{2}}{4+\pi^2} + \left(3 + \frac{2\pi}{4+\pi^2}\right)e^{-t}.$$

[If you are interested, the solution looks like this...]



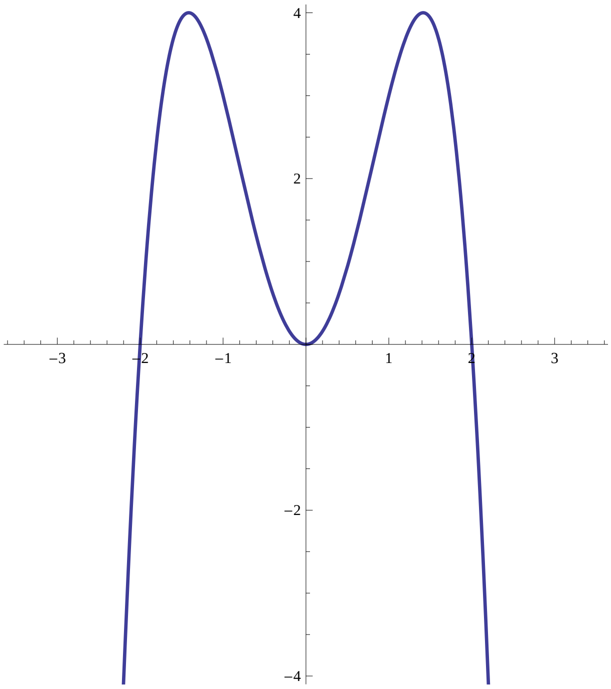
Soru 3 (Autonomous Equations)**.** Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y) = y^2(4 - y^2). \tag{4}$$

(a) [10p] Find all of the critical points of (4).

$y = -2, 0, 2$

(b) [15p] Sketch the graph of $f(y)$ versus y.



(c) [9p] Determine whether each critical point is asymptotically stable, unstable or semistable.

$y = -2$ is unstable $\boxed{3}$, $y = 0$ is semistable $\boxed{3}$, and $y = 2$ is asymptotically stable $\boxed{3}$.

(d) [16p] Sketch 10 (or more) different solutions of (4).