2016.01.05 **MAT371 Diferansiyel Denklemler – Final Sınavın Çözümleri** N. Course

Soru 1 (Carbon monoxide pollution).

English

The conference room, in the Engineering and Architecture Faculty, contains 4500 litres of air. Initially this air is free of carbon monoxide (CO).

Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 litres/minute.

A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 litres/minute.

(a) [18p] Write an initial value problem (IVP) for the amount of carbon monoxide in the conference room at time t.
(You must explain why your differential equation is valid.)

(b) [7p] Approximately what percentage, of the air in the conference room, will be carbon monoxide after one year (assuming that nothing changes in this year, the door isn't opened, etc.)?

Türkçe

Mühendislik ve Mimarlık Fakültesi'ndeki konferans salonu 4500 litre hava barındırmaktadır. Başlangıçta bu havada hiç karbon monoksit (CO) bulunmamaktadır.

t = 0 zamanından başlayarak, %4 karbon monoksit ihtiva eden sigara dumanı, odaya 0.3 litre/dakika oranında üflenmektedir.

Bir tavan vantilatörü odadaki havayı eşit olarak dağıtmaktadır ve odadan 0.3 litre/dakika oranında hava çıkmaktadır.

(a) [18p] Konferans salonundaki karbon monoksit miktarı için (t zamanda) bir başlangıç değer problemi (IVP) yazınız.
(Diferansiyel denkleminizin neden geçerli olduğunu açıklamalısınız.)

(b) [7p] Bir yılın sonunda bu konferans salonundaki havanın yaklaşık yüzde kaçı karbon monoksit olacaktır (bu bir yıl içinde hiç bir değişiklik olmadığını, örneğin kapının açılmadığını, vs. varsayarak)?

This is the easiest question on this is exam. I hope that you choose this one.
(a) Let t denote time measured in minutes. Let $c(t)$ denote the amount of carbon monoxide, in litres, in the conference room at time t. $\boxed{\text{3 : we must always say which units we are using!}}$ We know that $c(0) = 0$ litres.

Clearly, we have 0.04×0.3 litres/minute $= 0.012$ litres/minute of carbon monoxide entering the conference room.

The concentration of carbon monoxide in the room will be $\frac{c(t)}{4500}$. So $\frac{c(t)}{4500} \times 0.3 = \frac{0.0002c}{3}$ litres/minute of carbon monoxide will leave the room
Hence, our IVP is

$$\begin{cases} \frac{dc}{dt} = 0.012 - \frac{0.0002c}{3} = 0.0001(120 - \frac{2}{3}c), \\ a(0) = 0. \end{cases}$$

(b) Since t is measured in minutes, one year is a very long time. We might as well assume that $t = \infty$. After one year, we expect the concentration of carbon monoxide to have stabilised at an equilibrium solution.
Setting $\frac{dc}{dt} = 0$ we obtain $c = 180$ litres. So the concentration of carbon monoxide should be approximately $\frac{180}{4500} = 4\%$.

Soru 2 (Second Order Linear Differential Equations). [25p] Solve

$$2y'' + 8y' + 8y = 8 + 250t\cos t. \tag{1}$$

First consider the homogeneous equation $2y'' + 8y' + 8y = 0$. The characteristic equation is $2r^2 + 8r + 8 = 0$ which has repeated root $r = -2$. Therefore the general solution to $2y'' + 8y' + 8y = 0$ is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Next consider $2y'' + 8y' + 8y = 8$. It is easy to see that $Y(t) = 1$ is a particular solution to this ODE.

Now consider $2y'' + 8y' + 8y = 250t\cos t$. We try the ansatz $Y(t) = (At + B)\cos t + (Ct + D)\sin t$ and find that

$$\begin{aligned}
250t\cos t &= 2Y'' + 8Y' + 8Y \\
&= 2\Big((2C - B - At)\cos t + (-2A - Ct_D)B\sin t\Big) \\
&\quad + 8\Big((A + Ct + D)\cos t + (C - At - B)\sin t\Big) \\
&\quad + 8\Big((At + B)\cos t + (Ct + D)\sin t\Big) \\
&= \Big[6At + 8Ct + 8A + 6B + 4C + 8D\Big]\cos t \\
&\quad + \Big[-8At + 6Ct - 4A - 8B + 8C + 6D\Big]\sin t
\end{aligned}$$

Thus, we need $6A + 8C = 250$, $-8A + 6C = 0$, $8A + 6B + 4C + 8D = 0$ and $-4A - 8B + 8C + 6D = 0$. So we choose $A = 15$, $B = -4$, $C = 20$ and $D = -22$.
Hence

$$Y(t) = 15t\cos t - 4\cos t + 20t\sin t - 22\sin t.$$

Finally we add these 3 solutions together. Therefore, the general solution to the ODE is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + 1 + 15t\cos t - 4\cos t + 20t\sin t - 22\sin t.$$

Soru 3 (Systems of Equations).

(a) [13p] Solve

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}$ are $r_1 = -1$ and $r_2 = 3$. The corresponding eigenvectors are $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

Therefore, the general solution of $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}$ is

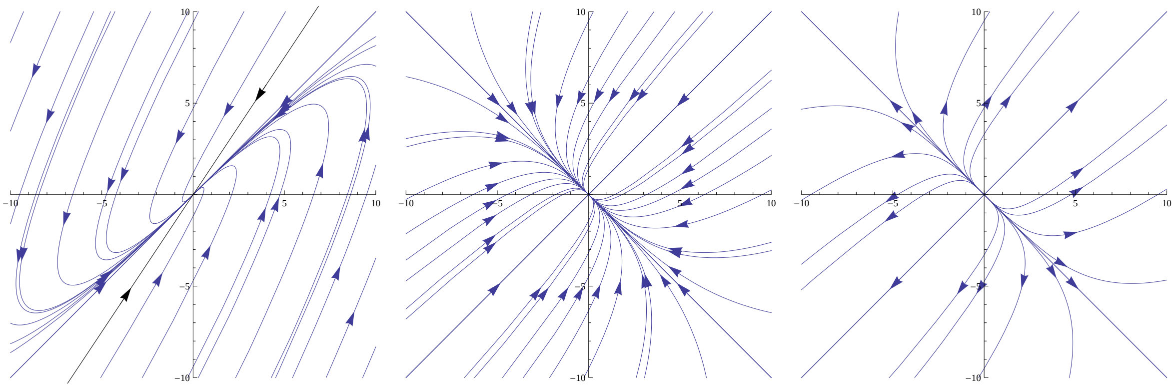
$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Finally, we use the initial condition to calculate that

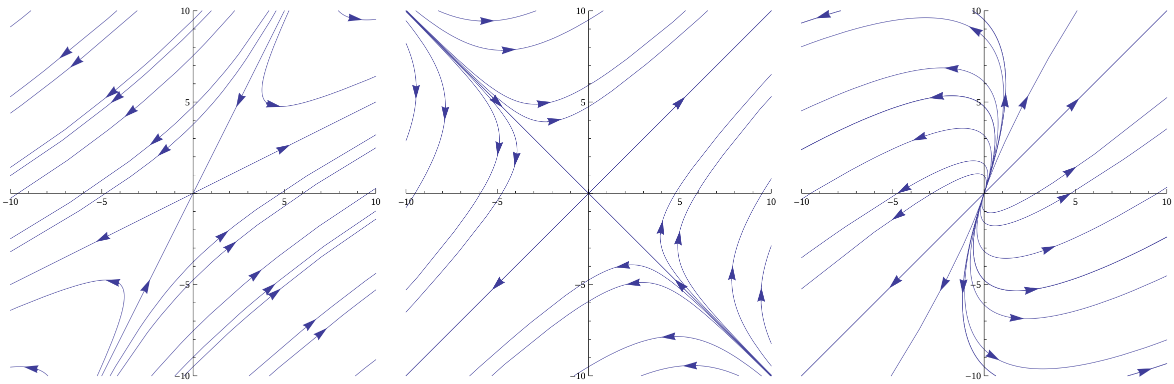
$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 + 5c_2 \end{pmatrix}$$

which tells us that $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$. Thus, the answer to this question is

$$\mathbf{x}(t) = \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$



(i) Stable node (ii) Stable node (iii) Unstable node



(iv) Saddle (v) Saddle (vi) Unstable node

Let $A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$. The determinant of A is 3 and the trace of A is -4. The eigenvalues of A are $r_1 = -3$ and $r_2 = -1$. The corresponding eigenvectors of A are $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively. A is a symmetric invertible 2×2 matrix.

(b) [2p] Which of the graphs (above) is the phase plot of the equation $\mathbf{x}' = A\mathbf{x}$?

[Mark ☑ one box only.]

☐ (i) ☑ (ii) ☐ (iii) ☐ (iv) ☐ (v) ☐ (vi)

(c) [10p] Justify (explain) your answer to part (b).

Since both eigenvalues are negative, we must have a phase plot with a stable node at the origin. So the phase plot must be either (i) or (ii).

The phase plot must also have straight lines in the directions of the eigenvectors. So it must be (ii).

Soru 4 (First Order Exact Equations). Consider

$$\left(\frac{1}{x} + \frac{\cosh(xy)}{x^2} + \frac{y}{x}\sinh(xy)\right) + \sinh(xy)\frac{dy}{dx} = 0 \tag{2}$$

This equation is of the form $M(x,y) + N(x,y)y' = 0$.

(a) [2p] Is this equation exact?

No, because

$$M_y = \frac{2}{x}\sinh(xy) + y\cosh(xy) \neq y\cosh(xy) = N_x.$$

(b) [2p] Calculate $\frac{M_y - N_x}{N}$.

$$\frac{M_y - N_x}{N} =$$

$$\frac{M_y - N_x}{N} = \frac{2}{x}$$

(c) [6p] Find an integrating factor $\mu(x)$ that solves

$$\frac{d\mu}{dx}(x) = \mu(x)\cdot\left(\frac{M_y - N_x}{N}\right)$$

$$\begin{aligned} \frac{d\mu}{dx} &= \frac{2\mu}{x} \\ \frac{d\mu}{\mu} &= 2\frac{dx}{x} \\ \log|\mu| &= 2\log|x| + C \\ \mu(x) &= cx^2. \end{aligned}$$

We can choose $c = 1$ to get $\mu(x) = x^2$.

(d) [1p] Multiply (2) by the integrating factor that you found in part (c).

$$\big(x + \cosh(xy) + xy\sinh(xy)\big) + x^2\sinh(xy)\frac{dy}{dx} = 0 \tag{3}$$

(e) [2p] Show that (3) is exact?

[HINT: Equation (3) is your answer to part (d). If (3) is not exact, then your answer to part (c) is probably wrong.]

$$\begin{aligned} M_y &= x\sinh(xy) + x\sinh(xy) + x^2y\cosh(xy) = 2x\sinh(xy) + x^2y\cosh(xy) \\ N_x &= 2x\sinh(xy) + x^2y\cosh(xy) = M_y \end{aligned}$$

Therefore (3) is exact.

(f) [12p] Solve (3).

We need to find a function $\psi(x,y)$ such that

$$\psi_x = x + \cosh(xy) + xy\sinh(xy)$$
$$\psi_y = x^2\sinh(xy).$$

Integrating the first equation wrt x gives

$$\psi(x,y) = \frac{1}{2}x^2 + x\cosh(xy) + h(y)$$

for some function $h(y)$. Then we differentiate wrt y to see that

$$\psi_y = x^2\sinh(xy) + h'(y).$$

We can choose $h(y) = 0$ to satisfy the ψ_y equation above. Therefore, the solution to (3) is

$$\frac{1}{2}x^2 + x\cosh(xy) = c$$

for some constant c.

Soru 5 (Second Order Exact Equations). The second order ordinary differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{4}$$

is said to be *exact* if and only if it can be written in the form

$$\big(P(x)y'\big)' + \big(f(x)y\big)' = 0 \tag{5}$$

for some function $f(x)$. Equation (5) can be integrated once immediately, resulting in a first order linear equation for y which you know how to solve.

(a) [20p] Show that

$$(4) \text{ is exact} \qquad \iff \qquad P''(x) - Q'(x) + R(x) = 0$$

[HINT: Start with $P(x)y'' + Q(x)y' + R(x)y = \big(P(x)y'\big)' + \big(f(x)y\big)'$. Equation (4) is exact iff this is true. Don't forget to prove both " $\Longleftarrow$ " and " $\Longrightarrow$ ".]

Equation (4) is exact iff

$$\begin{aligned} P(x)y'' + Q(x)y' + R(x)y &= \big(P(x)y'\big)' + \big(f(x)y\big)' \\ &= P(x)y'' + P'(x)y' + f(x)y' + f'(x)y \end{aligned}$$

which is true iff $f'(x) = R(x)$ and $P'(x) + f(x) = Q(x)$. Differentiating the second equation, we get

$$0 = P''(x) - Q(x) + f'(x) = P''(x) - Q'(x) + R(x).$$

Conversely, if $P''(x) - Q'(x) + R(x) = 0$ is true, then by setting setting $f(x) := Q(x) - P'(x)$, we obtain $f'(x) = R(x)$.

(b) [5p] Show that

$$y'' + xy' + y = 0$$

is exact.

And a very very easy 5 points to finish with:

$$0 = y'' + xy' + y = (y')' + (xy)'$$