OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

SON TESLİM TARİHİ: Çarşamba 13 Nisan 2016 saat 12:00'e kadar.

Egzersiz 10 (Separation of Variables). Consider

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X'(L) = 0. \end{cases}$$

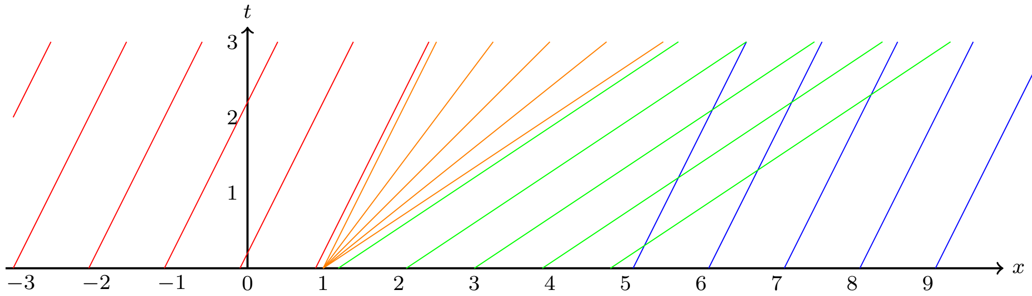
(a) [40p] Find all the eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions, if $\lambda < 0$.

(b) [20p] Find all the eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions, if $\lambda = 0$.

(c) [40p] Find all the eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions, if $\lambda > 0$.

9. (a) $\begin{cases} \frac{du}{dt} = 0 \\ \frac{dx}{dt} = \frac{u}{2} \end{cases}$

(b)



(c) The solution of the first ODE (away from the shock) is $u(x,t) = u(x_0, 0)$. Then $\frac{dx}{dt} = \frac{1}{2}u = \frac{1}{2}u(x_0, 0)$ has solution $x = \frac{1}{2}u(x_0,0)t + x_0 = \begin{cases} \frac{1}{2}t + x_0 & x_0 < 1 \\ \frac{3}{2}t + x_0 & 1 < x_0 < 5 \\ \frac{1}{2}t + x_0 & x_0 > 5 \end{cases}$.

We can see from (b) that there is a shock wave starting at $x_0 = 5$. Since $[u] = \lim_{x \searrow 5} u(x,0) - \lim_{x \nearrow 5} u(x,0) = 1 - 3 = -2$, $q(u) = \frac{1}{4}u^2$ (because $\frac{dq}{du} = \frac{1}{2}u$) and $[q] = \lim_{x \searrow 5} q(u(x,0)) - \lim_{x \nearrow 5} q(u(x,0)) = \frac{1}{4} \cdot 1^2 - \frac{1}{4} \cdot 3^2 = -2$, the shock characteristic is obtained by solving $\frac{dx_s}{dt} = \frac{[q]}{[u]} = 1$. So $x_s = t + x_s(0) = t + 5$. This is where the behaviour of the solution changes.

A priori, we know that the solution is

$$u(x,t) = \begin{cases} 1 & x < \textit{something} \\ \textit{something} & \textit{something} < x < \textit{something} \\ 3 & \textit{something} < x < 5 + t \\ 1 & x > 5 + t. \end{cases}$$

Next we look at the fan-like characteristics starting at $x_0 = 1$. From the characteristics, we can see that $x_0 < 1 \iff x - \frac{1}{2}t < 1 \iff x < \frac{1}{2}t + 1$ and $x_0 > 1 \iff x - \frac{3}{2}t > 1 \iff x > \frac{3}{2}t + 1$. Therefore

$$u(x,t) = \begin{cases} 1 & x < \frac{1}{2}t + 1 \\ \textit{something} & \frac{1}{2}t + 1 < x < \frac{3}{2}t + 1 \\ 3 & \frac{3}{2}t + 1 < x < 5 + t \\ 1 & x > 5 + t. \end{cases}$$

To complete our solution, we use the equation $x(t) = \frac{1}{2}ut + x_0$ with $x_0 = 1$. Rearranging gives $u = \frac{2x-2}{t}$. Therefore

$$u(x,t) = \begin{cases} 1 & x < \frac{1}{2}t + 1 \\ \frac{2x-2}{t} & \frac{1}{2}t + 1 < x < \frac{3}{2}t + 1 \\ 3 & \frac{3}{2}t + 1 < x < 5 + t \\ 1 & x > 5 + t. \end{cases}$$

(d)