2015–16 **MAT372 K.T.D.D. – Extra Problems** N. Course

!!! This is not homework. / Bu ödev değil. !!!

Problem 15 (Fourier Transforms). Use a Fourier Transform to solve

$$\begin{cases} u_t = ku_{xx} + cu_x, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x). \end{cases}$$

Problem 16 (Fourier Transforms). Use a Fourier Transform to solve

$$\begin{cases} u_{tt} = u_{xx}, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = 0, \\ u_t(x,0) = g(x). \end{cases}$$

Ödev 8'in çözümleri

14. (a) Using integration by parts, we calculate that $\mathcal{F}[f_x](\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(x,t)e^{-i\omega x}dx = -\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x,t)\frac{\partial}{\partial x}e^{-i\omega x}dx = \frac{i\omega}{2\pi}\int_{-\infty}^{\infty} f(x,t)e^{-i\omega x}dx = i\omega\mathcal{F}[f](\omega,t)$.

    (b) $\mathcal{F}\left[\frac{\partial^2 f}{\partial x^2}\right] = i\omega\mathcal{F}\left[\frac{\partial f}{\partial x}\right] = (i\omega)^2\mathcal{F}[f] = -\omega^2\mathcal{F}[f]$.

    (c) $\mathcal{F}[g](\omega) = \frac{\sin\omega a}{\pi\omega}$

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15. Taking Fourier Transforms gives

$$\begin{cases} U_t = -k\omega^2 U + ic\omega U, & t > 0, \\ U(\omega,0) = F(\omega). \end{cases}$$

which has solution $U(\omega,t) = F(\omega)e^{-(k\omega^2 - ic\omega)t}$. Next we use the inverse Fourier Transform to see that

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} F(\omega)e^{-(k\omega^2-ic\omega)t}e^{i\omega x}\, d\omega = \int_{-\infty}^{\infty}\left(e^{i\omega ct}F(\omega)\right)\left(e^{-k\omega^2 t}\right)e^{i\omega x}\, d\omega \\ &= \int_{-\infty}^{\infty} H(\omega)G(\omega)e^{i\omega x}\, d\omega = h(x,t) * g(x,t) \end{aligned}$$

where $h = \mathcal{F}^{-1}[H(\omega)] = \mathcal{F}^{-1}[e^{i\omega ct}F(\omega)] = f(x+ct)$ and $g = \mathcal{F}^{-1}[G(\omega)] = \mathcal{F}^{-1}[e^{-k\omega^2 t}] = \sqrt{\frac{\pi}{kt}}e^{-\frac{x^2}{4kt}}$. Therefore

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi+ct)\sqrt{\frac{\pi}{kt}}e^{-\frac{(x-\xi)^2}{4kt}}\, d\xi$$

16. Taking Fourier Transforms gives

$$\begin{cases} U_{tt} = -\omega^2 U, \\ U(\omega,0) = 0 \\ U_t(\omega,0) = G(\omega). \end{cases}$$

which has solution $U(\omega,t) = A(\omega)\cos\omega t + B(\omega)\sin\omega t = (0)\cos\omega t + \left(\frac{G(\omega)}{\omega}\right)\sin\omega t = \frac{G(\omega)}{\omega}\sin\omega t$. Using the inverse Fourier Transform, we see that $u(x,t) = \int_{-\infty}^{\infty} G(\omega)\frac{\sin\omega t}{\omega}e^{i\omega x}\, d\omega = g(x) * f(x,t)$ where $f(x,t) = \begin{cases} 0 & |x| > t \\ \pi & |x| < t. \end{cases}$

Therefore, the solution is

$$u(x,t) = \frac{1}{2}\int_{-t}^{t} g(x-\xi)\, d\xi.$$