OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

24.05.2012 **MAT 372 – K.T.D.D. – Yarıyıl Sonu Sınavı** N. Course

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| ADI SOYADI |
| ÖĞRENCİ NO |
| İMZA |

**Do not open the next page until you are told that the exam has started.**

1. You will have 120 minutes to answer 4 questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. You should write your student number on every page.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the final 10 minutes of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam 120 dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan 4 tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4 sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Öğrenci numaranızı her sayfaya yazınız.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

**Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.**

| 1 | 2 | 3 | 4 | 5 | TOTAL |
|---|---|---|---|---|---|
| | | | | | |

## Canonical Forms:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G \\
H^* &= -D^*u_\xi - E^*u_\eta - F^*u + G^* \\
\frac{dy}{dx} &= \frac{B \pm \sqrt{\Delta}}{2A}
\end{aligned}$$

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## Fourier Transforms:

$$F(\omega) = \mathcal{F}\big[f\big](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\ dx$$

$$f(x) = \mathcal{F}^{-1}\big[F\big](x) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\ d\omega$$

| $f(x)$ | $F(\omega)$ |
|---|---|
| $u_t(x,t)$ | $U_t(\omega,t)$ |
| $u_x(x,t)$ | $i\omega U(\omega,t)$ |
| $u_{xx}(x,t)$ | $-\omega^2 U(\omega,t)$ |
| $e^{-\alpha x^2}$ | $\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\omega}}$ |
| $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\ d\xi$ | $F(\omega)G(\omega)$ |
| $\delta(x-x_0)$ | $\frac{1}{2\pi}e^{-i\omega x_0}$ |
| $f(x-\beta)$ | $e^{-i\omega\beta}F(\omega)$ |
| $xf(x)$ | $iF_\omega(\omega)$ |
| $\frac{2\alpha}{x^2+\alpha^2}$ | $e^{-\lvert\omega\rvert\alpha}$ |
| $f(x) = \begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$ | $\frac{\sin a\omega}{\pi\omega}$ |

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## Famous PDEs:

| | |
|---|---|
| $u_t = ku_{xx}$ | heat equation |
| $u_{tt} - c^2u_{xx} = 0$ | wave equation |
| $\nabla^2 u = 0$ | Laplace's Equation |

## Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{L}x$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi}{L}x\ dx$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi}{L}x\ dx$$

If $f(x) = \sum_{k=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ then

$$f'(x) = \sum_{k=1}^{\infty} -\left(\frac{n\pi}{L}\right)a_n \sin\frac{n\pi x}{L}.$$

If $f(x) = \sum_{k=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ then

$$\begin{aligned}
f'(x) &= \frac{1}{L}\Big[f(L) - f(0)\Big] \\
&\quad + \sum_{k=1}^{\infty}\left[\frac{n\pi}{L}b_n + \frac{2}{L}\Big((-1)^n f(L) - f(0)\Big)\right]\cos\frac{n\pi x}{L}.
\end{aligned}$$

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## ODEs:

The solution of $\phi' = \mu\phi$ is

$$\phi(x) = Ae^{\mu x}.$$

The solution of $\phi'' = \mu^2\phi$ is

$$\begin{aligned}
\phi(x) &= Ae^{\mu x} + Be^{-\mu x} \\
&= C\cosh\mu x + D\sinh\mu x.
\end{aligned}$$

The solution of $\phi'' = -\mu^2\phi$ is

$$\phi(x) = A\cos\mu x + B\sin\mu x.$$

The solution of $x(x\phi')' - \mu^2\phi = 0$ ($\mu \neq 0$) is

$$\phi(x) = Ax^{-\mu} + Bx^{\mu}.$$

The solution of $x(x\phi')' = 0$ is

$$\phi(x) = A\log x + B.$$

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**Question 1** (Fourier Transforms). Consider the Wave Equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,0) = f(x) \\ u_t(x,0) = 0. \end{cases} \tag{1}$$

(a) [5 pts] If $\mathcal{F}$ denotes the Fourier Transform operator with respect to $x$, show that

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t}\mathcal{F}[u] \qquad \text{and} \qquad \mathcal{F}\left[\frac{\partial u}{\partial x}\right] = i\omega \mathcal{F}[u].$$

(b) [2 pts] Deduce that

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2}{\partial t^2}\mathcal{F}[u] \qquad \text{and} \qquad \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -\omega^2 \mathcal{F}[u].$$

(c) [5 pts] Let $U = \mathcal{F}[u]$ and $F = \mathcal{F}[f]$. Use the formulae in part (b) to take Fourier Transforms of equation (1).

$$\begin{cases} \\ \\ \end{cases}$$

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(d) [5 pts] Solve the boundary value problem for $U$ [that you wrote in part (c)] and show that

$$U(\omega, t) = \frac{1}{2} F(\omega) \left( e^{ic\omega t} + e^{-ic\omega t} \right).$$

[HINT: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$.]

(e) [8 pts] Use the Inverse Fourier Transform, $\mathcal{F}^{-1}$, to show that

$$u(x, t) = \frac{1}{2} \Big( f(x + ct) + f(x - ct) \Big)$$

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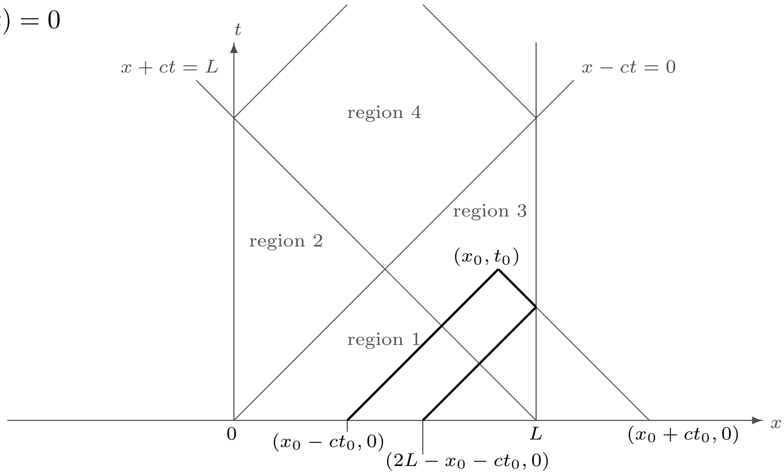
**Question 2** (Finite String Wave Equation)**.** Consider the wave equation on a string of length $L$ with fixed ends:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \qquad t > 0 \\ u(x,0) = f(x) & f : (0,L) \to \mathbb{R} \\ u_t(x,0) = g(x) & g : (0,L) \to \mathbb{R} \\ u(0,t) = 0 & \\ u(L,t) = 0 & \end{cases} \tag{2}$$

where $c > 0$.




Let

$$\text{region 3} := \{(x,t) : x \leq L, x - ct \geq 0 \text{ and } x + ct \geq L\}.$$

In this question, you will calculate the solution in region 3.

(a) [5 pts] First show that

$$u(x,t) = F(x - ct) + G(x + ct)$$

solves the wave equation, $u_{tt} - c^2 u_{xx} = 0$, for any twice differentiable functions $F : (0,L) \to \mathbb{R}$ and $G : (0,L) \to \mathbb{R}$.

Using the initial conditions we can see that:

$$\begin{aligned} f(x) &= u(x,0) = F(x) + G(x) \\ g(x) &= u_t(x,0) = -cF'(x) + cG'(x) \end{aligned} \tag{3}$$

(b) [5 pts] Use (3) to show that

$$-F(x) + G(x) = \frac{1}{c} \int_0^x g(z)\ dz.$$

[HINT: You may assume that $F(0) = G(0)$]

(c) [4 pts] Use (b) and (3) to show that

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\ dz.$$

(d) [4 pts] Next use (a) and (2) show that

$$G(L + ct) = -F(L - ct).$$

and that

$$G(z) = -F(2L - z) \qquad \text{for all } z \geq L.$$

(e) [7 pts] Use (a), (c) and (d) to show that the solution in region 3 is

$$u(x,t) = \frac{f(x - ct) - f(2L - x - ct)}{2} - \frac{1}{2c}\int_0^{x-ct} g(\xi)\ d\xi + \frac{1}{2c}\int_0^{2L-x-ct} g(\xi)\ d\xi. \qquad (4)$$

**Question 3** (Characteristics)**.** Consider the PDE

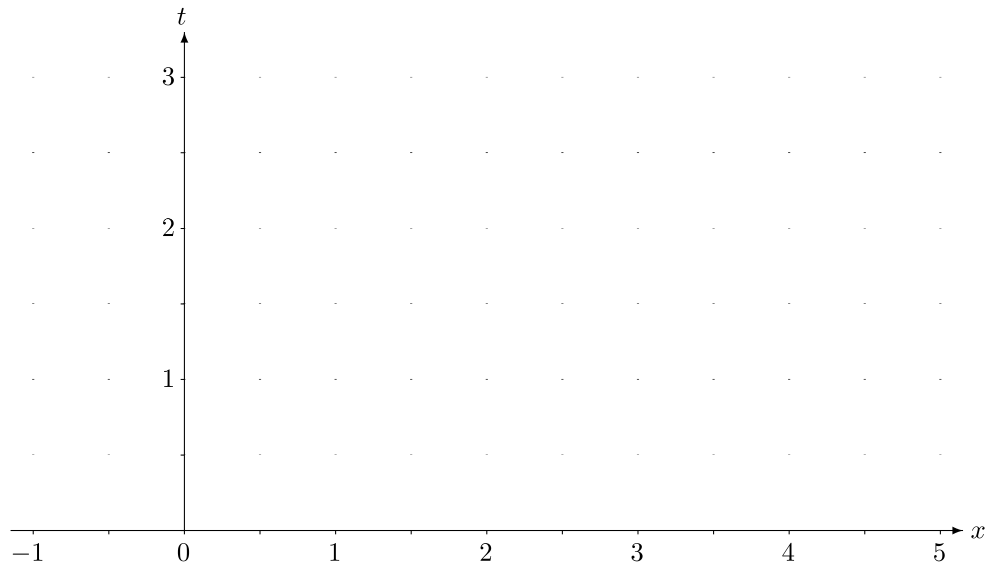
$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \tag{5}$$

with the initial condition

$$u(x,0) = \begin{cases} 3 & x < 2 \\ 1 & x > 2. \end{cases} \tag{6}$$

(a) [3 pts] Replace (5) by a system of 2 ODEs

(b) [6 pts] Plot the characteristics ($t$ against $x$) for this problem.




(c) [1 pts] Does the problem have *fan-like characteristics* or *shock wave characteristics*?

☐ fan-like characteristics ☐ shock wave characteristics ☐ neither

[Mark ☑ only one box.]

(d) [10 pts] Solve

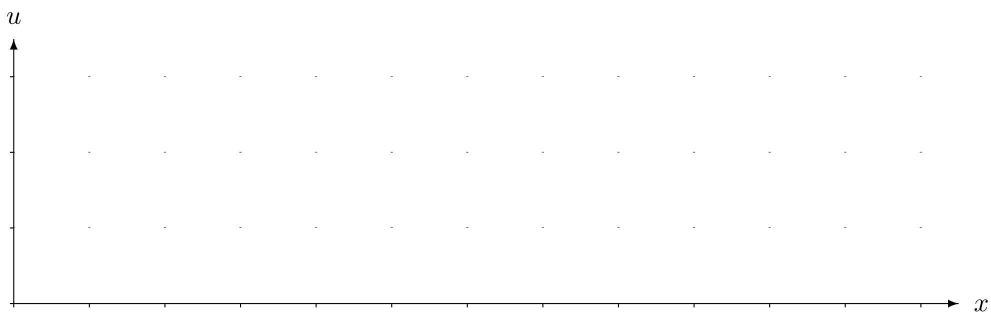
$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

subject to

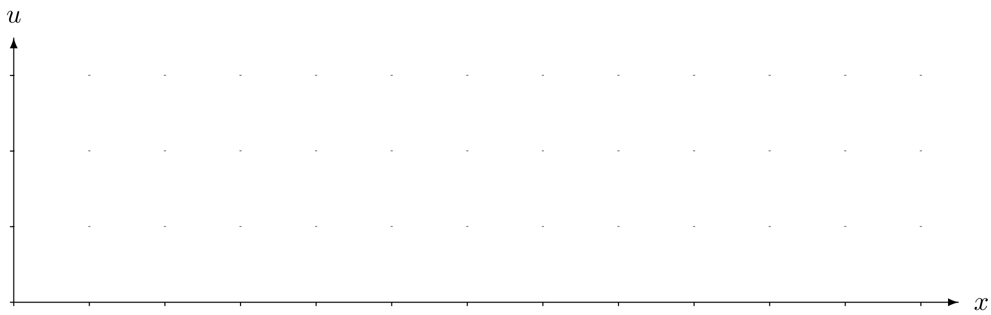
$$u(x,0) = \begin{cases} 3 & x < 2 \\ 1 & x > 2. \end{cases}$$

$$u(x,t) = \begin{cases} \quad & \text{if } x < \\ \quad & \text{if } x > \end{cases}$$

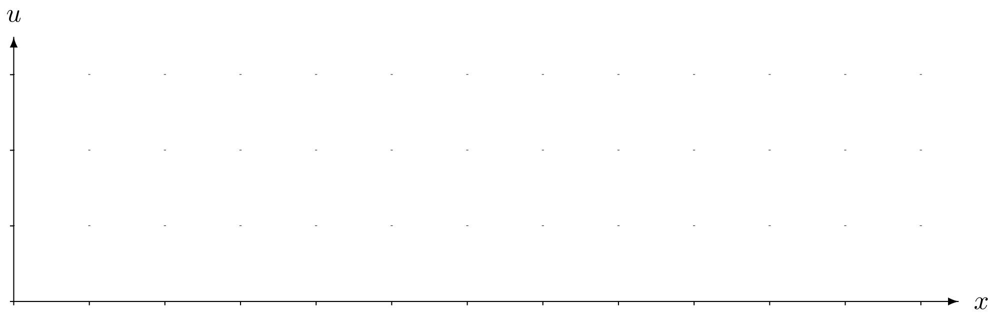
(e) [5 pts] Sketch the graph ($u$ against $x$) of the solution at times $t = 0$, $t = 1$ and $t = 2$.




($t = 0$)




($t = 1$)




($t = 2$)

**Question 4** (Separation of Variables)**.** Consider the heat equation on a rod of length $L$:

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u_x(L,t) = 0 \\ u(x,0) = 7 - \cos\frac{3\pi x}{L}. \end{cases} \tag{7}$$

(a) [5 pts] If $u(x,t) = X(x)T(t)$, show that $X$ and $T$ satisfy

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + k\lambda T = 0$$

for some constant $\lambda \in \mathbb{R}$.

(b) [3 pts] What boundary conditions does $X$ satisfy?

$$\begin{cases} \\ \\ \end{cases}$$

(c) [10 pts] By considering the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, find all the eigenvalues and eigenfunctions of

$$X'' + \lambda X = 0$$

subject to the boundary conditions that you wrote in part (b).

(d) [4 pts] Find the general solution of

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u_x(L,t) = 0. \end{cases}$$

(e) [3 pts] Now use the initial condition,

$$u(x,0) = 7 - \cos \frac{3\pi x}{L},$$

to write down the solution to equation (7).

**Question 5** (Fourier Series)**.** Define the function $f : [-1, 1] \to \mathbb{R}$ by

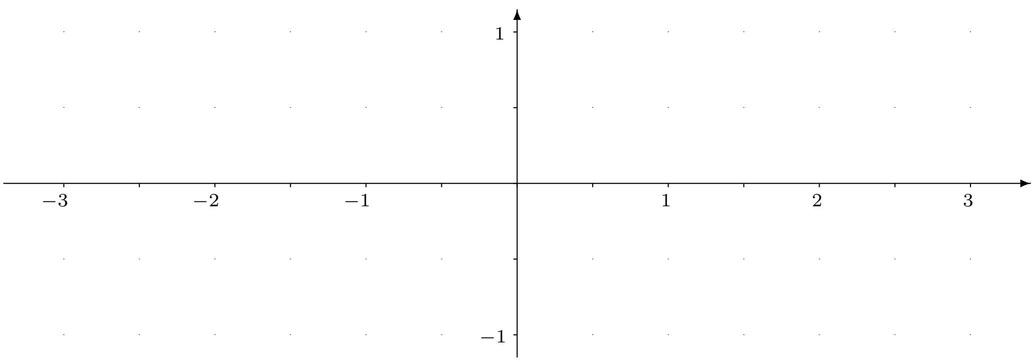
$$f(x) = x \tag{8}$$

(a) [6 pts] Show that

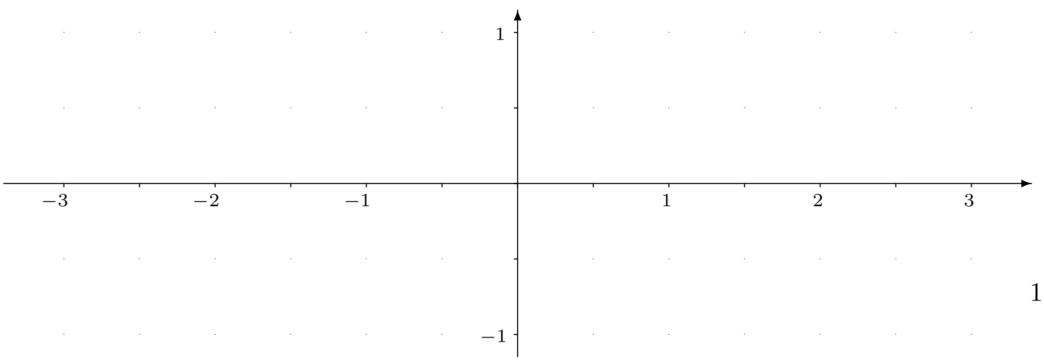
$$\{\cos n\pi x : n \in \mathbb{N}\}$$

is an orthogonal system on $[-1, 1]$ with respect to the weight function $w(x) = 1$.

(b) [2 pts] Sketch $f$.




(c) [5 pts] Sketch the Fourier Series of $f$.



(d) [12 pts] Calculate the coefficients ($a_0$, $a_k$ and $b_k$, for $k = 1, 2, 3, \ldots$) of the Fourier Series of $f(x) = x$.