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MATEMATİK BÖLÜMÜ

24.05.2012 **MAT 372 – K.T.D.D. – Yarıyıl Sonu Sınavı Çözümleri** N. Course

Question 1 (Fourier Transforms)**.** Consider the Wave Equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,0) = f(x) \\ u_t(x,0) = 0. \end{cases} \tag{1}$$

(a) [5 pts] If $\mathcal{F}$ denotes the Fourier Transform operator with respect to x, show that

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t}\mathcal{F}[u] \qquad \text{and} \qquad \mathcal{F}\left[\frac{\partial u}{\partial x}\right] = i\omega\mathcal{F}[u].$$

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right](\omega,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}(x,t)e^{-i\omega x}dx = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx\right) = \frac{\partial}{\partial t}\mathcal{F}[u](\omega,t)$$

and

$$\begin{aligned}\mathcal{F}\left[\frac{\partial u}{\partial x}\right](\omega,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}(x,t)e^{-i\omega x}dx = -\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x,t)\frac{\partial}{\partial x}\left(e^{-i\omega x}\right)dx \\ &= \frac{i\omega}{2\pi}\int_{-\infty}^{\infty}u(x,t)e^{-i\omega x}dx = i\omega\mathcal{F}[u](\omega,t)\end{aligned}$$

by integration by parts.

(b) [2 pts] Deduce that

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial^2}{\partial t^2}\mathcal{F}[u] \qquad \text{and} \qquad \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -\omega^2\mathcal{F}[u].$$

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial}{\partial t}\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial^2}{\partial t^2}\mathcal{F}[u],$$

and

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = i\omega\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = (i\omega)^2\mathcal{F}[u] = -\omega^2\mathcal{F}[u].$$

(c) [5 pts] Let $U = \mathcal{F}[u]$ and $F = \mathcal{F}[f]$. Use the formulae in part (b) to take Fourier Transforms of equation (1).

$$\begin{cases} U_{tt} + c^2\omega^2 U = 0 \\ U(\omega,0) = F(\omega) \\ U_t(\omega,0) = 0 \end{cases}$$

(d) [5 pts] Solve the boundary value problem for U [that you wrote in part (c)] and show that

$$U(\omega,t) = \frac{1}{2}F(\omega)\left(e^{ic\omega t} + e^{-ic\omega t}\right).$$

[HINT: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$.]

The general solution of $U_{tt} + c^2\omega^2 U = 0$ is $U(\omega,t) = A(\omega)\cos c\omega t + B(\omega)\sin c\omega t)$. Then

$$0 = U_t(\omega,t) = -c\omega A(\omega)\sin c\omega 0 + c\omega B(\omega)\cos c\omega 0 = -c\omega B(\omega) \implies B(\omega) = 0\ \forall\omega\in\mathbb{R},$$

and

$$F(\omega) = U(\omega,0) = A(\omega)\cos c\omega 0 = A(\omega).$$

So

$$U(\omega,t) = F(\omega)\cos c\omega t = F(\omega)\left(\frac{e^{ic\omega t} + e^{-ic\omega t}}{2}\right).$$

(e) [8 pts] Use the Inverse Fourier Transform, $\mathcal{F}^{-1}$, to show that

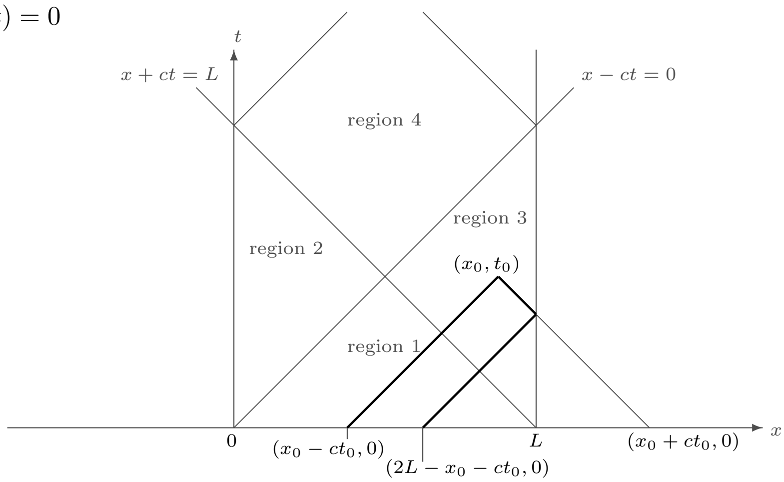
$$u(x,t) = \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big).$$

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}[U](x,t)\\
&= \int_{-\infty}^{\infty} U(\omega,t)e^{i\omega x}d\omega\\
&= \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)\left(e^{ic\omega t} + e^{-ic\omega t}\right)e^{i\omega x}d\omega\\
&= \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)e^{i\omega(x+ct)}d\omega + \frac{1}{2}\int_{-\infty}^{\infty} F(\omega)e^{i\omega(x-ct)}d\omega\\
&= \frac{1}{2}\mathcal{F}^{-1}[F](x+ct) + \frac{1}{2}\mathcal{F}^{-1}[F](x-ct)\\
&= \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big)
\end{aligned}$$

Question 2 (Finite String Wave Equation)**.** Consider the wave equation on a string of length L with fixed ends:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \quad t > 0\\ u(x,0) = f(x) & f:(0,L)\to\mathbb{R}\\ u_t(x,0) = g(x) & g:(0,L)\to\mathbb{R}\\ u(0,t) = 0 & \\ u(L,t) = 0 & \end{cases} \tag{2}$$

where $c > 0$.



Let

$$\text{region } 3 := \{(x,t) : x \le L, x - ct \ge 0 \text{ and } x + ct \ge L\}.$$

In this question, you will calculate the solution in region 3.

(a) [5 pts] First show that

$$u(x,t) = F(x-ct) + G(x+ct)$$

solves the wave equation, $u_{tt} - c^2 u_{xx} = 0$, for any twice differentiable functions $F : (0,L) \to \mathbb{R}$ and $G : (0,L) \to \mathbb{R}$.

Since $u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$, $u_{tt}(x,t) = c^2F''(x-ct) + c^2G''(x+ct)$, $u_x(x,t) = F'(x-ct) + G'(x+ct)$ and $u_{xx}(x,t) = F''(x-ct) + G''(x+ct)$, we have that

$$u_{tt} - c^2 u_{xx} = (c^2F'' + c^2G'') - c^2(F'' + G'') = 0.$$

Using the initial conditions we can see that:

$$\begin{aligned} f(x) &= u(x,0) = F(x) + G(x) \\ g(x) &= u_t(x,0) = -cF'(x) + cG'(x) \end{aligned} \tag{3}$$

(b) [5 pts] Use (3) to show that

$$-F(x) + G(x) = \frac{1}{c}\int_0^x g(z)\ dz.$$

[HINT: You may assume that $F(0) = G(0)$]

$$\begin{aligned} \int_0^x g(z)dz &= \int_0^x -cF'(z) + cG'(z)dz = c\Big[-F(z) + G(z)\Big]_0^x \\ &= c\Big(-F(x) + G(x) + F(0) - G(0)\Big) = c(-F(x) + G(x)). \end{aligned}$$

Therefore

$$-F(x) + G(x) = \frac{1}{c}\int_0^x g(z)\ dz.$$

(c) [4 pts] Use (b) and (3) to show that

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\ dz.$$

$$\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\ dz = \frac{1}{2}\big(F(x) + G(x)\big) - \frac{1}{2}\big(-F(x) + G(x)\big) = F(x).$$

(d) [4 pts] Next use (a) and (2) show that

$$G(L+ct) = -F(L-ct).$$

and that

$$G(z) = -F(2L - z) \qquad \text{for all } z \ge L.$$

$$0 = u(L,t) = F(L-ct) + G(L+ct) \implies G(L+ct) = -F(L-ct). \ \boxed{2}$$

For all $z \ge L$, let $t = \frac{1}{c}(z - L) \ge 0$. Then $L + ct = z$ and so

$$G(z) = G(L+ct) = -F(L-ct) = -F(L-(z-L)) = -F(2L-z). \ \boxed{2}$$

(e) [7 pts] Use (a), (c) and (d) to show that the solution in region 3 is

$$u(x,t) = \frac{f(x-ct) - f(2L-x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} g(\xi)\, d\xi + \frac{1}{2c}\int_0^{2L-x-ct} g(\xi)\, d\xi. \tag{4}$$

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= F(x-ct) - F(2L-(x+ct)) \\ &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x+ct} g(z)dz - \frac{1}{2}f(2L-x-ct) + \frac{1}{2c}\int_0^{2L-x-ct} g(z)dz \\ &= \frac{f(x-ct) - f(2L-x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} g(\xi)\, d\xi + \frac{1}{2c}\int_0^{2L-x-ct} g(\xi)\, d\xi. \end{aligned}$$

Question 3 (Characteristics)**.** Consider the PDE

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \tag{5}$$

with the initial condition

$$u(x,0) = \begin{cases} 3 & x < 2 \\ 1 & x > 2. \end{cases} \tag{6}$$

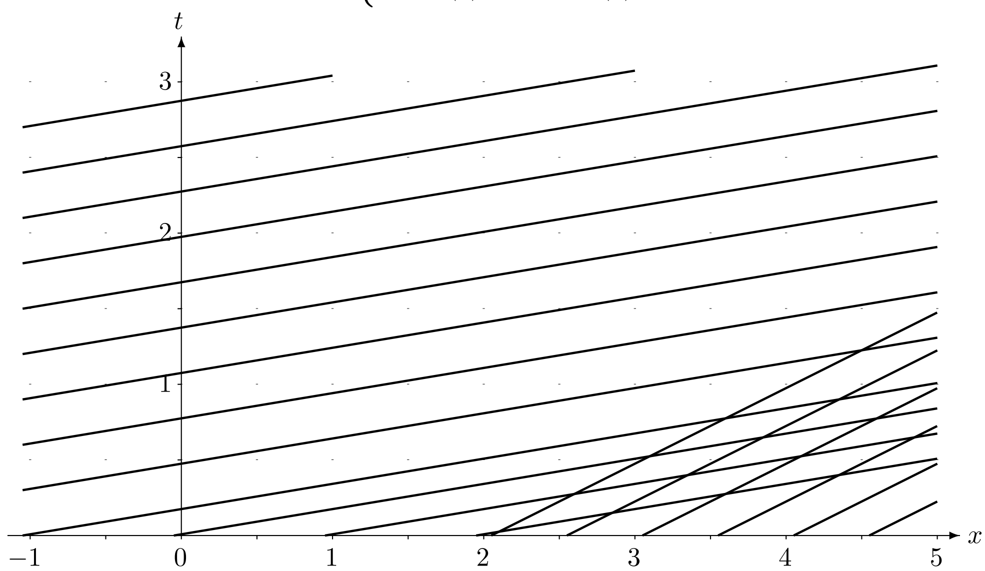
(a) [3 pts] Replace (5) by a system of 2 ODEs

$$\begin{cases} \frac{du}{dt} = 0 \\ \frac{dx}{dt} = 2u \end{cases}$$

(b) [6 pts] Plot the characteristics (t against x) for this problem.

First $u_t = 0 \implies u(x(t),t) = u(x(0),0)\ \forall t$. Then $\frac{dx}{dt} = 2u \implies x(t) = 2ut + x(0)$. So

$$x(t) = \begin{cases} 6t + x(0) & \text{if } x(0) < 2 \\ 2t + x(0) & \text{if } x(0) > 2. \end{cases}$$



(c) [1 pts] Does the problem have *fan-like characteristics* or *shock wave characteristics*?

☐ fan-like characteristics ☑ shock wave characteristics ☐ neither

(d) [10 pts] Solve

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

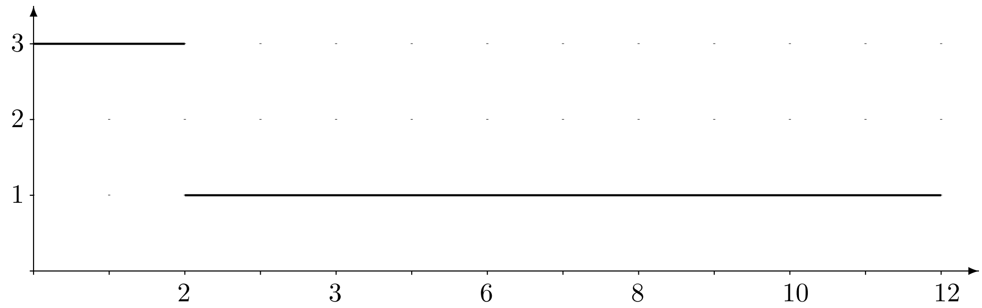
subject to

$$u(x,0) = \begin{cases} 3 & x < 2 \\ 1 & x > 2. \end{cases}$$

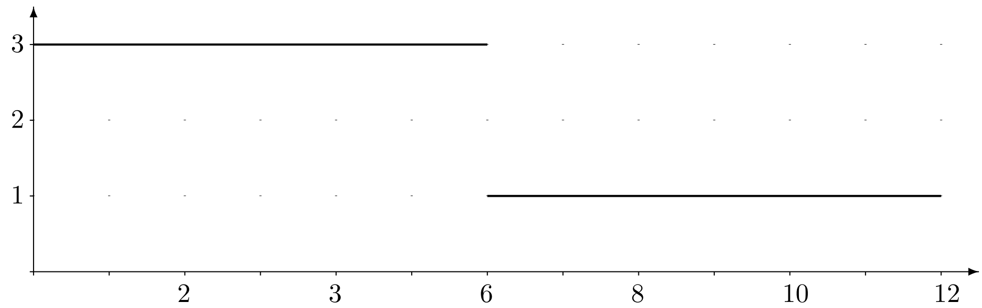
At the discontinuity ($x = 2$), we have $[u] = \lim_{x \searrow 2} u(x,0) - \lim_{x \nearrow 2} u(x,0) = 1 - 3 = -2$. Let $q(u) = u^2$. (Then $\frac{dq}{du} = 2u$.) Then $[q] = \lim_{x \searrow 2} q(u(x,0)) - \lim_{x \nearrow 2} 1(u(x,0)) = 1^2 - 3^2 = -8$. The shock characteristic is found by solving $\frac{dx_s}{dt} = \frac{[q]}{[u]} = \frac{-8}{-2} = 4$. So $x_s(t) = 4t + x_s(0) = 4t + 2$.
Therefore

$$\boxed{u(x,t) = \begin{cases} 3 & \text{if } x < 4t + 2 \\ 1 & \text{if } x > 4t + 2. \end{cases}}$$

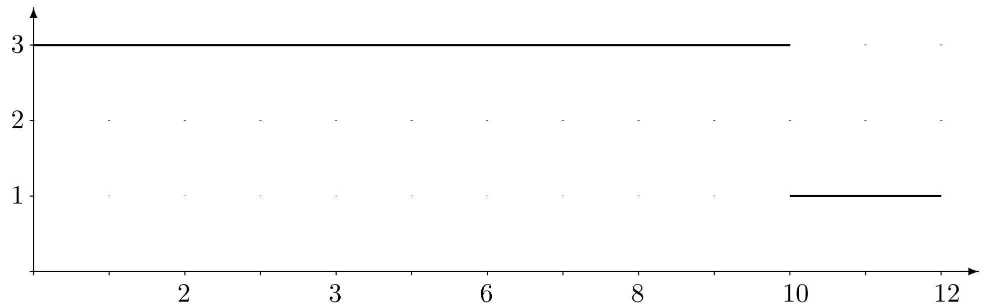
(e) [5 pts] Sketch the graph (u against x) of the solution at times $t = 0$, $t = 1$ and $t = 2$.



$(t = 0)$



$(t = 1)$



$(t = 2)$

Question 4 (Separation of Variables)**.** Consider the heat equation on a rod of length L:

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u_x(L,t) = 0 \\ u(x,0) = 7 - \cos\frac{3\pi x}{L}. \end{cases} \tag{7}$$

(a) [5 pts] If $u(x,t) = X(x)T(t)$, show that X and T satisfy

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + k\lambda T = 0$$

for some constant $\lambda \in \mathbb{R}$.

Since $XT' - u_t = ku_{xx} = kX''T$, we have that $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$ $\boxed{2}$. The left-hand side is a function only of x; the right-hand side is a function only of t. Therefore both sides must be equal to a constant; equal to $-\lambda$ say $\boxed{2}$. Then $\frac{X''}{X} = -\lambda \implies X'' + \lambda X = 0$ and $\frac{T'}{kT} = -\lambda \implies T' + k\lambda T = 0$ $\boxed{1}$.

(b) [3 pts] What boundary conditions does X satisfy?

First note that $0 = u_x(0,t) = X'(0)T(t)$ and $0 = u_x(L,t) = X'(L)T(t)$. Since we don't want $T(t) = 0\ \forall t$, we must have... $\boxed{\text{optional}}$

$$\begin{cases} X'(0) = 0 \\ X'(L) = 0 \end{cases}$$

(c) [10 pts] By considering the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, find all the eigenvalues and eigenfunctions of

$$X'' + \lambda X = 0$$

subject to the boundary conditions that you wrote in part (b).

CASE 1: $\lambda < 0$.
The solution of $X'' + \lambda X = 0$ is $X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$. Then $0 = X'(0) = A\sqrt{-\lambda}e^0 - B\sqrt{-\lambda}e^0 \implies A = B$ and $0 = X'(L) = A\sqrt{-\lambda}(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}) \implies A = 0 \implies B = 0$. There are no eigenvalues or non-trivial eigenfunctions in this case. $\boxed{2}$

CASE 2: $\lambda = 0$.
The solution of $X'' = 0$ is $X(x) = Ax + B$. Then $0 = X'(0) = A$ and $0 = X'(L) = A \implies A = 0$. We can choose any B we like. Therefore $\lambda_0 = 0$ is an eigenvalue with eigenfunction $X_0(x) = 1$. $\boxed{4}$

CASE 3: $\lambda > 0$.
The solution of $X'' + \lambda X = 0$ is $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. So $0 = X'(0) = -A\sqrt{\lambda}\sin\sqrt{\lambda}0 + B\sqrt{\lambda}\cos\sqrt{\lambda}0 \implies B = 0$; and $0 = X'(L) = -A\sqrt{\lambda}\sin\sqrt{\lambda}L$. Since we don't want $A = 0$, we must have that $\sin\sqrt{\lambda}L = 0$. So $\sqrt{\lambda}L = n\pi$, $n = 1,2,3,\ldots$. So $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ are eigenvalues with eigenfunctions $X_n(x) = \cos\frac{n\pi x}{L}$. $\boxed{4}$

(d) [4 pts] Find the general solution of

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u_x(L,t) = 0. \end{cases}$$

The solution of $T_n' + k\lambda_n T_n = 0$ is $T_n(t) = a_n e^{-k\lambda_n t}$. So

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L}.$$

(e) [3 pts] Now use the initial condition,

$$u(x,0) = 7 - \cos\frac{3\pi x}{L},$$

to write down the solution to equation (7).

Clearly $a_0 = 7$, $a_3 = -1$ and $a_n = 0$ for all other n. Therefore

$$u(x,t) = 7 - e^{-k\left(\frac{3\pi}{L}\right)^2 t} \cos\frac{3\pi x}{L}.$$

Question 5 (Fourier Series)**.** Define the function $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = x \tag{8}$$

(a) [6 pts] Show that

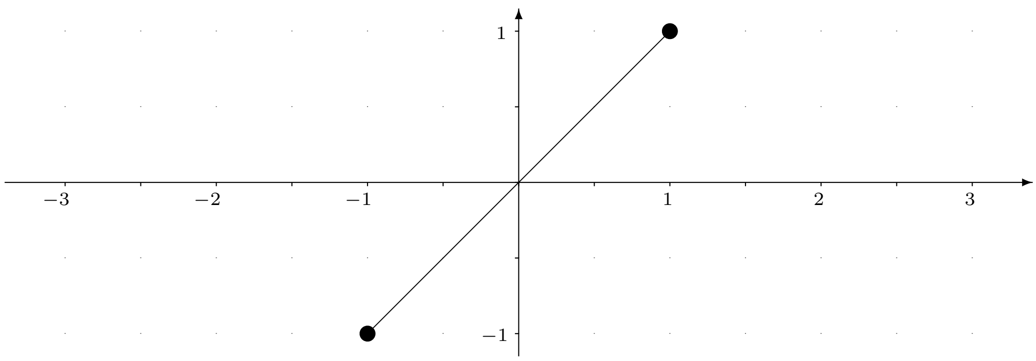
$$\{\cos n\pi x : n \in \mathbb{N}\}$$

is an orthogonal system on $[-1,1]$ with respect to the weight function $w(x) = 1$.

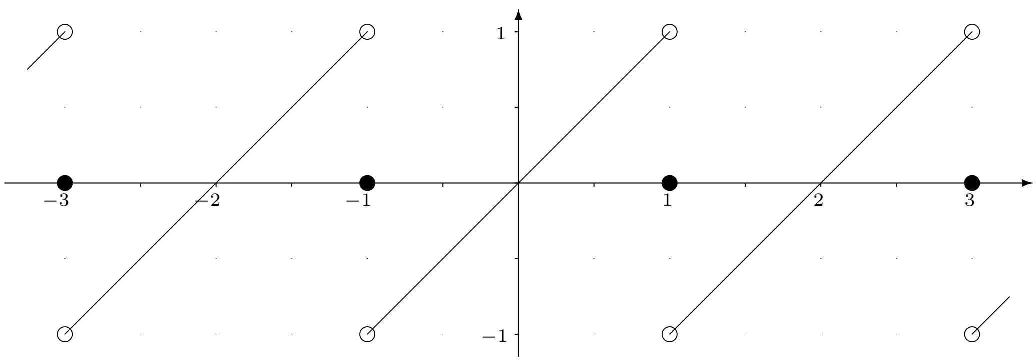
Let $n \neq m$. Since $\cos(A-B) + \cos(A+B) = 2\cos A \cos B$,

$$\begin{aligned}
\langle \cos n\pi x, \cos m\pi x\rangle_1 &= \int_{-1}^{1} \cos n\pi x \cos m\pi x \; dx \\
&= \frac{1}{2}\int_{-1}^{1} \cos(n-m)\pi x + \cos(n+m)\pi x \; dx \\
&= \frac{1}{2}\left[\frac{1}{(n-m)\pi}\sin(n-m)\pi x + \frac{1}{(n+m)\pi}\sin(n+m\pi x\right]_{-1}^{1} \\
&= 0
\end{aligned}$$

(b) [2 pts] Sketch f.



(c) [5 pts] Sketch the Fourier Series of f.



(d) [12 pts] Calculate the coefficients (a_0, a_k and b_k, for $k = 1, 2, 3, \dots$) of the Fourier Series of $f(x) = x$.

First

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} x\ dx = 0\ \boxed{3}$$

because $f(x) = x$ is an odd function. Similarly $x \cos k\pi x$ is an odd function, so

$$a_k = \int_{-1}^{1} x \cos k\pi x\ dx = 0\ \forall k \in \mathbb{N}\ \boxed{3}.$$

Finally,

$$\begin{aligned}
b_k &= \int_{-1}^{1} x \sin k\pi x\ dx \\
&= \left[\frac{-x\cos k\pi x}{k\pi}\right]_{-1}^{1} + \int_{-1}^{1} \frac{\cos k\pi x}{k\pi} dx \qquad \text{(integration by parts)} \\
&= \left[\frac{-x\cos k\pi x}{k\pi} + \frac{\sin k\pi x}{k^2\pi^2}\right]_{-1}^{1} \\
&= -\frac{\cos k\pi}{k\pi} + 0 - \frac{(-1)\cos k\pi(-1)}{k\pi} - 0 \\
&= \frac{-2\cos k\pi}{k\pi} \\
&= \frac{-2}{k\pi}(-1)^k.\ \boxed{6}
\end{aligned}$$