OKAN ÜNİVERSİTESİ
FEN EDEBİYAT FAKÜLTESİ
MATEMATİK BÖLÜMÜ

2013.05.22 **MAT 372 – K.T.D.D. – Final Sınavı** N. Course

Adı: Ö R N E K T İ R

Soyadı: S A M P L E

Öğrenci No: 0 1 0 6 0

İmza:

Süre: **120** dk.

Bu sorulardan **4** tanesini seçerek cevaplayınız

**Do not open the exam until you are told that you may begin.**
**Sınavın başladığı yüksek sesle söylenene kadar sayfayı çevirmeyin.**

1. You will have **120** minutes to answer **4** questions from a choice of 5. If you choose to answer more than 4 questions, then only your best 4 answers will be counted.
2. The points awarded for each part, of each question, are stated next to it.
3. All of the questions are in English. You may answer in English or in Turkish.
4. Write your student number on every page.
5. If you wish to leave before the end of the exam, give your exam script to an invigilator and leave the room quietly. You may not leave in the final 10 minutes of the exam.
6. Calculators, mobile phones and any digital means of communication are forbidden. The sharing of pens, erasers or any other item between students is forbidden.
7. All bags, coats, books, notes, etc. must be placed away from your desks and away from the seats next to you. You may not access these during the exam. Take out everything that you will need before the exam starts.
8. Any student found cheating or attempting to cheat will receive a mark of zero (0), and will be investigated according to the regulations of Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği.

1. Sınav süresi toplam **120** dakikadır. Sınavda 5 soru sorulmuştur. Bu sorulardan **4** tanesini seçerek cevaplayınız. 4'den fazla soruyu cevaplarsanız, en yüksek puanı aldığınız 4sorunun cevapları geçerli olacaktır.
2. Soruların her bölümünün kaç puan olduğu yanlarında belirtilmiştir.
3. Tüm sorular İngilizce'dir. Cevaplarınızı İngilizce yada Türkçe verebilirsiniz.
4. Öğrenci numaranızı her sayfaya yazınız.
5. Sınav süresi sona ermeden sınavınızı teslim edip çıkmak isterseniz, sınav kağıdınızı gözetmenlerden birine veriniz ve sınav salonundan sessizce çıkınız. Sınavın son 10 dakikası içinde sınav salonundan çıkmanız yasaktır.
6. Sınav esnasında hesap makinesi, cep telefonu ve dijital bilgi alışverişi yapılan her türlü malzemelerin kullanımı ile diğer silgi, kalem, vb. alışverişlerin yapılması kesinlikle yasaktır.
7. Çanta, palto, kitap ve ders notlarınız gibi eşyalarınız sıraların üzerinden ve yanınızdaki sandalyeden kaldırılmalıdır. Sınav süresince bu tür eşyaları kullanmanız yasaktır, bu nedenle ihtiyacınız olacak herşeyi sınav başlamadan yanınıza alınız.
8. Her türlü sınav, ve diğer çalışmada, kopya çeken veya kopya çekme girişiminde bulunan bir öğrenci, o sınav ya da çalışmadan sıfır (0) not almış sayılır, ve o öğrenci hakkında Yükseköğretim Kurumları Öğrenci Disiplin Yönetmeliği hükümleri uyarınca disiplin kovuşturması yapılır.

| 1 | 2 | 3 | 4 | 5 | Toplam |
|---|---|---|---|---|---|
| | | | | | |

Canonical Forms:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G \\
H^* &= -D^*u_\xi - E^*u_\eta - F^*u + G^* \\
\frac{dy}{dx} &= \frac{B \pm \sqrt{\Delta}}{2A}
\end{aligned}$$

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Fourier Transforms:

$$F(\omega) = \mathcal{F}\big[f\big](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\ dx$$

$$f(x) = \mathcal{F}^{-1}\big[F\big](x) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\ d\omega$$

| $f(x)$ | $F(\omega)$ |
|---|---|
| $u_t(x,t)$ | $U_t(\omega,t)$ |
| $u_x(x,t)$ | $i\omega U(\omega,t)$ |
| $u_{xx}(x,t)$ | $-\omega^2 U(\omega,t)$ |
| $e^{-\alpha x^2}$ | $\frac{1}{\sqrt{4\pi\alpha}}e^{-\frac{\omega^2}{4\omega}}$ |
| $\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\ d\xi$ | $F(\omega)G(\omega)$ |
| $\delta(x-x_0)$ | $\frac{1}{2\pi}e^{-i\omega x_0}$ |
| $f(x-\beta)$ | $e^{-i\omega\beta}F(\omega)$ |
| $xf(x)$ | $iF_\omega(\omega)$ |
| $\frac{2\alpha}{x^2+\alpha^2}$ | $e^{-\lvert\omega\rvert\alpha}$ |
| $f(x) = \begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$ | $\frac{\sin a\omega}{\pi\omega}$ |

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Famous PDEs:

| | |
|---|---|
| $u_t = ku_{xx}$ | heat equation |
| $u_{tt} - c^2u_{xx} = 0$ | wave equation |
| $\nabla^2 u = 0$ | Laplace's Equation |

Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\frac{k\pi}{L}x + b_k\sin\frac{k\pi}{L}x$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx$$

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi}{L}x\ dx$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi}{L}x\ dx$$

If $f(x) = \sum_{k=1}^{\infty} a_n\cos\frac{n\pi x}{L}$ then

$$f'(x) = \sum_{k=1}^{\infty} -\left(\frac{n\pi}{L}\right)a_n\sin\frac{n\pi x}{L}.$$

If $f(x) = \sum_{k=1}^{\infty} b_n\sin\frac{n\pi x}{L}$ then

$$\begin{aligned}
f'(x) &= \frac{1}{L}\Big[f(L) - f(0)\Big] \\
&\quad + \sum_{k=1}^{\infty}\left[\frac{n\pi}{L}b_n + \frac{2}{L}\Big((-1)^n f(L) - f(0)\Big)\right]\cos\frac{n\pi x}{L}.
\end{aligned}$$

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ODEs:

The solution of $\phi' = \mu\phi$ is

$$\phi(x) = Ae^{\mu x}.$$

The solution of $\phi'' = \mu^2\phi$ is

$$\begin{aligned}
\phi(x) &= Ae^{\mu x} + Be^{-\mu x} \\
&= C\cosh\mu x + D\sinh\mu x.
\end{aligned}$$

The solution of $\phi'' = -\mu^2\phi$ is

$$\phi(x) = A\cos\mu x + B\sin\mu x.$$

The solution of $x(x\phi')' - \mu^2\phi = 0$ $(\mu \neq 0)$ is

$$\phi(x) = Ax^{-\mu} + Bx^{\mu}.$$

The solution of $x(x\phi')' = 0$ is

$$\phi(x) = A\log x + B.$$

**Soru 1** (Canonical Forms)**.** Consider the second order partial differential equation

$$(\sin^2 x)u_{xx} + (\sin 2x)u_{xy} + (\cos^2 x)u_{yy} = x. \tag{1}$$

(a) [1p] Calculate the discriminant $\Delta(x, y)$ of (1).

(b) [2p] Equation (1) is a

☐ hyperbolic PDE; ☐ parabolic PDE; ☐ elliptic PDE.

(c) [2p] Find the characteristic equation of (1).

(d) [5p] Find the characteristic curve(s) of (1).
[HINT: $\int \sec z \; dz = \log|\sec z + \tan z| + c$, $\int \cot z \; dz = \log|\sin z| + c$ and $\int \operatorname{cosec} z \; dz = -\log|\operatorname{cosec} z + \cot z| + c$]

$$(\sin^2 x)u_{xx} + +(\sin 2x)u_{xy} + (\cos^2 x)u_{yy} = x \qquad (1)$$

(e) [15p] Find a canonical form for (1).

[HINT: $x$ and $y$ MUST NOT appear in your answer, I only want to see $u$, $\xi$ and $\eta$.]

[HINT: $\cos^2 z = 1 - \sin^2 z$.]

**Soru 2** (Ortogonality).

(a) [3p] Let $f, g : [\alpha, \beta] \to \mathbb{R}$ be 2 piecewise continuous functions. Give the definition of the *inner product* of $f$ and $g$ on $[\alpha, \beta]$.

[HINT: Repeating students should assume that the weighting function is $w(x) \equiv 1$ – in other words; give the definition of the $L^2$-inner product on $[\alpha, \beta]$. $3^{rd}$ year students can ignore this comment.]

(b) [6p] Show that the inner product satisfies the following conditions for all piecewise continuous functions $f, g, h : [\alpha, \beta] \to \mathbb{R}$ and for all $\lambda, \mu \in \mathbb{R}$:

(i) $\langle f, f \rangle \geq 0$;

(ii) $\langle f, g \rangle = \langle g, f \rangle$;

(iii) $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$; and

(iv) $\langle f, \lambda g + \mu h \rangle = \lambda \langle f, g \rangle + \mu \langle f, h \rangle$.

(c) [2p] Give the definition of an *orthogonal system* of functions on $[\alpha, \beta]$.

Let $L > 0$ and define $\phi_n : [-L, L] \to \mathbb{R}$ by

$$\phi_n := \cos \frac{n\pi x}{L}. \tag{2}$$

(d) [14p] Show that $\{\phi_n\}_{n=1}^{\infty}$ is an orthogonal system on $[-L, L]$.
[HINT: $\cos(A + B) + \cos(A - B) =?$ and $\cos(A + B) - \cos(A - B) =?$]

**Soru 3** (Integration of Fourier Series)**.** Let $L > 0$ and define $f : [0, L] \to \mathbb{R}$ by $f(x) \equiv 1$.

(a) [10p] Calculate the Fourier Sine Series of $f$ on $[0, L]$.

[HINT: "Fourier Sine Series" means that you should have $a_n = 0$ for all $n = 0, 1, 2, 3, \dots$]

[HINT: The formulae on page 2 are for a function $g : [-L, L] \to \mathbb{R}$, so don't just copy them blindly: Do you remember how we changed the formulae to calculate the Fourier Sine/Cosine series of a function $g : [0, L] \to \mathbb{R}$?]

(b) [10p] By integrating your answer to (a), show that

$$x \sim -\frac{4}{\pi} \sum_{k=1,3,5,\dots} \frac{1}{k^2\pi} \cos\frac{k\pi x}{L} + \frac{4L}{\pi}\left(\frac{1}{\pi} + \frac{1}{3^2\pi} + \frac{1}{5^2\pi} + \frac{1}{7^2\pi} + \dots\right)$$

[HINT: $\int_0^x 1\ dt = [t]_0^x = x$]

(c) [5p] Given that the Fourier Cosine Series for $x$ is

$$x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

where $a_0 = \frac{2}{L} \int_0^L x \, dx$, show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{8}. \tag{3}$$

**Soru 4** (Separation of Variables)**.** Consider the heat equation on a rod of length $L$:

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = h(x). \end{cases} \tag{4}$$

(a) [5p] If $u(x,t) = X(x)T(t)$, show that $X$ and $T$ satisfy

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + k\lambda T = 0$$

for some constant $\lambda \in \mathbb{R}$.

(b) [3p] What boundary conditions does $X$ satisfy?

$$\begin{cases} \\ \\ \end{cases}$$

(c) [12p] By considering the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, find all the eigenvalues and eigenfunctions of

$$X'' + \lambda X = 0$$

subject to the boundary conditions that you wrote in part (b).

(d) [5p] Find the general solution of

$$\begin{cases} u_t = k u_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 & \\ u(L,t) = 0. & \end{cases}$$

**Soru 5** (Fourier Transforms). [25p] Use the Fourier Transform to solve

$$\begin{cases} u_t = ku_{xxx}, & -\infty < x < \infty, \quad 0 < y < \infty, \\ u(x,0) = f(x). \end{cases} \tag{5}$$

[HINT: You may give your answer as a double integral, or as a convolution of 2 functions.]