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MATEMATİK BÖLÜMÜ

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Question 1 (Canonical Forms)**.** Consider the second order partial differential equation

$$(\sin^2 x)u_{xx} + (\sin 2x)u_{xy} + (\cos^2 x)u_{yy} = x. \tag{1}$$

(a) [1p] Calculate the discriminant $\Delta(x, y)$ of (1).

$$\Delta = B^2 - 4AC = 0$$

(b) [2p] Equation (1) is a

☐ hyperbolic PDE; ☑ parabolic PDE; ☐ elliptic PDE.

(c) [2p] Find the characteristic equation of (1).

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \cot x$$

(d) [5p] Find the characteristic curve(s) of (1).
[HINT: $\int \sec z \; dz = \log|\sec z + \tan z| + c$, $\int \cot z \; dz = \log|\sin z| + c$ and $\int \operatorname{cosec} z \; dz = -\log|\operatorname{cosec} z + \cot z| + c$]

$$y = \log \sin x + c$$

$$(\sin^2 x)u_{xx} + +(\sin 2x)u_{xy} + (\cos^2 x)u_{yy} = x \tag{1}$$

(e) [15p] Find a canonical form for (1).
[HINT: x and y MUST NOT appear in your answer, I only want to see u, ξ and η.]
[HINT: $\cos^2 z = 1 - \sin^2 z$.]

Letting $\xi = y - \log \sin x$ and $\eta = y$, we have $\boxed{3}$

$$\xi_x = -\cot x \qquad \xi_y = 1 \qquad \xi_{xx} = \frac{1}{\sin^2 x} \qquad \xi_{xy} = 0 \qquad \xi_{yy} = 0$$

$$\eta_x = 0 \qquad \xi_y = 1 \qquad \xi_{xx} = 0 \qquad \xi_{xy} = 0 \qquad \xi_{yy} = 0$$

Therefore $\boxed{3}$

$$A^* = B^* = 0$$
$$C^* = 0 + 0 + \cos^2 x \cdot 1 = \cos^2 x$$
$$D^* = \sin^2 x \left(\frac{1}{\sin^2 x}\right) + 0 + 0 + 0 + 0 = 1$$
$$E^* = 0 + 0 + 0 + 0 + 0 = 0$$
$$F^* = 0$$
$$G^* = x$$

Therefore $\boxed{3}$

$$\cos^2 x u_{\eta\eta} + u_\xi = x.$$

Finally, we calculate that $\log \sin x = y - \xi = \eta - \xi$. Therefore $\sin x = e^{\eta-\xi} \implies \cos^2 x = 1 - \sin^2 x = 1 - e^{2(\eta-\xi)}$ and $x = \sin^{-1} e^{\eta-\xi}$.
Hence, the answer $\boxed{6}$ is

$$\left(1 - e^{2(\eta-\xi)}\right) u_{\eta\eta} + u_\xi = \arcsin e^{\eta-\xi}.$$

Question 2 (Ortogonality).

(a) [3p] Let $f, g : [\alpha, \beta] \to \mathbb{R}$ be 2 piecewise continuous functions. Give the definition of the *inner product* of f and g on $[\alpha, \beta]$.
[HINT: Repeating students should assume that the weighting function is $w(x) \equiv 1$ – in other words; give the definition of the L^2-inner product on $[\alpha, \beta]$. 3^{rd} year students can ignore this comment.]

$$\langle f, g \rangle = \int_\alpha^\beta f(x)g(x)\, dx.$$

(b) [6p] Show that the inner product satisfies the following conditions for all piecewise continuous functions $f, g, h : [\alpha, \beta] \to \mathbb{R}$ and for all $\lambda, \mu \in \mathbb{R}$:

(a) $\langle f, f \rangle \geq 0$;
(b) $\langle f, g \rangle = \langle g, f \rangle$;
(c) $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$; and
(d) $\langle f, \lambda g + \mu h \rangle = \lambda \langle f, g \rangle + \mu \langle f, h \rangle$.

(a) Clearly $\langle f, f \rangle = \int_\alpha^\beta (f(x))^2\, dx \geq 0$. $\boxed{2}$

(b) That $\langle f, g \rangle = \int_\alpha^\beta f(x)g(x)\, dx = \int_\alpha^\beta g(x)f(x)\, dx = \langle g, f \rangle$ is trivial. $\boxed{1}$

(c) $\langle \lambda f + \mu g, h \rangle = \int_\alpha^\beta (\lambda f(x) + \mu g(x))h(x)\, dx = \lambda \int_\alpha^\beta f(x)h(x)\, dx + \mu \int_\alpha^\beta g(x)h(x)\, dx = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$. $\boxed{2}$

(d) Follows immediately from (ii) and (iii). $\boxed{1}$

(c) [2p] Give the definition of an *orthogonal system* of functions on $[\alpha, \beta]$.

The set of functions $\{\phi_n\}$ is called an *orthogonal system* on $[\alpha, \beta]$ iff $\langle \phi_n, \phi_m \rangle = 0$ for all $n \neq m$.

Let $L > 0$ and define $\phi_n : [-L, L] \to \mathbb{R}$ by

$$\phi_n := \cos \frac{n\pi x}{L}. \tag{2}$$

(d) [14p] Show that $\{\phi_n\}_{n=1}^{\infty}$ is an orthogonal system on $[-L, L]$.
[HINT: $\cos(A+B)+\cos(A-B)=?$ and $\cos(A+B)-\cos(A-B)=?$]

For $n \neq m$, we calculate that

$$\begin{aligned}
\langle \phi_n, \phi_m \rangle &= \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \quad \boxed{2} \\
&= \int_{-L}^{L} \frac{1}{2}\cos\frac{(n+m)\pi x}{L} + \frac{1}{2}\cos\frac{(n-m)\pi x}{L}\,dx \quad \boxed{4} \\
&= \frac{1}{2}\left[\frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L} + \frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\right]_{-L}^{L} \quad \boxed{4} \\
&= 0. \quad \boxed{4}
\end{aligned}$$

Question 3 (Integration of Fourier Series). Let $L > 0$ and define $f : [0, L] \to \mathbb{R}$ by $f(x) \equiv 1$.

(a) [10p] Calculate the Fourier Sine Series of f on $[0, L]$.
[HINT: "Fourier Sine Series" means that you should have $a_n = 0$ for all $n = 0, 1, 2, 3, \ldots$]
[HINT: The formulae on page 2 are for a function $g : [-L, L] \to \mathbb{R}$, so don't just copy them blindly: Do you remember how we changed the formulae to calculate the Fourier Sine/Cosine series of a function $g : [0, L] \to \mathbb{R}$?]

$$\begin{aligned}
b_k &= \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx = \frac{2}{L}\int_0^L \sin\frac{k\pi x}{L}\,dx = \frac{2}{L}\left[-\frac{L}{k\pi}\cos\frac{k\pi x}{L}\right]_0^L \\
&= \frac{2}{k\pi}\left(1-(-1)^k\right) \quad \boxed{5}
\end{aligned}$$

Therefore

$$1 \sim \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\left(1-(-1)^k\right)}{k}\sin\frac{k\pi x}{L} = \frac{4}{\pi}\sum_{k=1,3,5,\ldots}\frac{1}{k}\sin\frac{k\pi x}{L} \quad \boxed{5}$$

(b) [10p] By integrating your answer to (a), show that

$$x \sim -\frac{4}{\pi}\sum_{k=1,3,5,\ldots}\frac{1}{k^2\pi}\cos\frac{k\pi x}{L} + \frac{4L}{\pi}\left(\frac{1}{\pi}+\frac{1}{3^2\pi}+\frac{1}{5^2\pi}+\frac{1}{7^2\pi}+\ldots\right)$$

[HINT: $\int_0^x 1\,dt = [t]_0^x = x$]

$$\begin{aligned}
x &\sim \int_0^x \frac{4}{\pi}\sum_{k=1,3,5,\ldots}\frac{1}{k}\sin\frac{k\pi t}{L}\,dt \quad \boxed{3} \\
&= \left[-\frac{4}{\pi}\sum_{k=1,3,5,\ldots}\frac{L}{k^2\pi}\cos\frac{k\pi t}{L}\right]_0^x \quad \boxed{2} \\
&= -\frac{4}{\pi}\sum_{k=1,3,5,\ldots}\frac{L}{k^2\pi}\cos\frac{k\pi x}{L} + \frac{4}{\pi}\sum_{k=1,3,5,\ldots}\frac{L}{k^2\pi}\cos\frac{0}{L} \quad \boxed{2} \\
&= -\frac{4}{\pi}\sum_{k=1,3,5,\ldots}\frac{1}{k^2\pi}\cos\frac{k\pi x}{L} + \frac{4L}{\pi}\left(\frac{1}{\pi}+\frac{1}{3^2\pi}+\frac{1}{5^2\pi}+\frac{1}{7^2\pi}+\ldots\right) \quad \boxed{3}.
\end{aligned}$$

(c) [5p] Given that the Fourier Cosine Series for x is

$$x \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$$

where $a_0 = \frac{2}{L}\int_0^L x\,dx$, show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots = \frac{\pi^2}{8}. \tag{3}$$

And an easy problem to finish with: Since $a_0 = \frac{2}{L}\int_0^L x\,dx = \frac{2}{L}\left[\frac{1}{2}x^2\right]_0^L = L$ $\boxed{1}$, we can see that

$$\frac{L}{2} = \frac{a_0}{2} = \frac{4L}{\pi}\left(\frac{1}{\pi} + \frac{1}{3^2\pi} + \frac{1}{5^2\pi} + \frac{1}{7^2\pi} + \dots\right). \ \boxed{3}$$

Rearranging gives the required formula. $\boxed{1}$

Question 4 (Separation of Variables)**.** Consider the heat equation on a rod of length L:

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = h(x). \end{cases} \tag{4}$$

(a) [5p] If $u(x,t) = X(x)T(t)$, show that X and T satisfy

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + k\lambda T = 0$$

for some constant $\lambda \in \mathbb{R}$.

Since $XT' - u_t = ku_{xx} = kX''T$, we have that $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$ $\boxed{2}$. The left-hand side is a function only of x; the right-hand side is a function only of t. Therefore both sides must be equal to a constant; equal to $-\lambda$ say $\boxed{2}$. Then $\frac{X''}{X} = -\lambda \implies X'' + \lambda X = 0$ and $\frac{T'}{kT} = -\lambda \implies T' + k\lambda T = 0$ $\boxed{1}$.

(b) [3p] What boundary conditions does X satisfy?

First note that $0 = u_x(0,t) = X'(0)T(t)$ and $0 = u_L, t) = X(L)T(t)$. Since we don't want $T(t) = 0\ \forall t$, we must have... $\boxed{\text{optional}}$

$$\begin{cases} X'(0) = 0 \ \boxed{1.5} \\ X(L) = 0. \ \boxed{1.5} \end{cases}$$

(c) [12p] By considering the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, find all the eigenvalues and eigenfunctions of

$$X'' + \lambda X = 0$$

subject to the boundary conditions that you wrote in part (b).

CASE 1: $\lambda < 0$.
The solution of $X'' + \lambda X = 0$ is $X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$. Then $0 = X'(0) = A\sqrt{-\lambda}e^0 - B\sqrt{-\lambda}e^0 \implies A = B$; and $0 = X(L) = A(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}) \implies A = 0 \implies B = 0$. There are no eigenvalues and no non-trivial eigenfunctions in this case. $\boxed{3}$

CASE 2: $\lambda = 0$.
The solution of $X'' = 0$ is $X(x) = Ax + B$. Then $0 = X'(0) = A$ and $0 = X(L) = AL + B \implies A = 0 = B$. There are no eigenvalues and no non-trivial eigenfunctions in this case. $\boxed{3}$

CASE 3: $\lambda > 0$.
The solution of $X'' + \lambda X = 0$ is $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. So $0 = X'(0) = -A\sqrt{\lambda}\sin\sqrt{\lambda}0 + B\sqrt{\lambda}\cos\sqrt{\lambda}0 = B\sqrt{\lambda} \implies B = 0$; and $0 = X(L) = A\cos\sqrt{\lambda}L$. Since we don't want $A = 0$, we must have that $\cos\sqrt{\lambda}L = 0$. So $\sqrt{\lambda}L = (n - \frac{1}{2})\pi$, $n = 1, 2, 3, \dots$. So $\lambda_n = \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2$ are eigenvalues, with eigenfunctions $X_n(x) = \cos\frac{(n-\frac{1}{2})\pi x}{L}$. $\boxed{6}$

(d) [5p] Find the general solution of

$$\begin{cases} u_t = k u_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u(L,t) = 0. \end{cases}$$

The solution of $T_n' + k\lambda_n T_n = 0$ is $T_n(t) = a_n e^{-k\lambda_n t}$ $\boxed{2}$. So

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2 t} \cos\frac{(n-\frac{1}{2})\pi x}{L} \quad \boxed{3}$$

for some constants a_n.

Question 5 (Fourier Transforms). [25p] Use the Fourier Transform to solve

$$\begin{cases} u_t = k u_{xxx}, & -\infty < x < \infty, \quad 0 < t < \infty, \\ u(x,0) = f(x). \end{cases} \tag{5}$$

[HINT: You may give your answer as a double integral, or as a convolution of 2 functions.]

Taking Fourier Transforms, the problem becomes

$$\begin{cases} U_t = (i\omega)^3 kU = -ik\omega^3 U \\ U(\omega,0) = F(x). \quad \boxed{5} \end{cases}$$

which has solution

$$U(\omega,t) = F(\omega)e^{-ik\omega^3 t}. \quad \boxed{5}$$

If we define $G(\omega) = e^{-ik\omega^3 t}$, then we have $U(\omega,t) = F(\omega)G(\omega,t)$. So $u(x,t) = g(x,t) * f(x)$ $\boxed{5}$ where

$$g(x,t) = \mathcal{F}^{-1}[G](x,t) = \int_{-\infty}^{\infty} e^{-ik\omega^3 t} e^{i\omega x} d\omega = \int_{-\infty}^{\infty} e^{-i(k\omega^3 t - \omega x)} d\omega. \quad \boxed{5}$$

Therefore

$$u(x,t) = g(x,t) * f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi) g(x-\xi,t)\, d\xi. \quad \boxed{5}$$