2013.04.04 **MAT 372 – K.T.D.D. – Ara Sınavın Çözümleri** N. Course

Question 1 (Infinite String Wave Equation)**.** Consider the wave equation on a string of infinite length:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty, \qquad t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x). \end{cases} \tag{1}$$

(a) [8p] Suppose that $F : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$ are twice differentiable functions (i.e. F', G', F'' and G'' all exist). Let $u(x,t) = F(x-ct) + G(x+ct)$. Show that

$$u_{tt} - c^2 u_{xx} = 0.$$

Since $u_x = F'(x-ct)+G'(x-ct)$, $u_{xx} = F''(x-ct)+G''(x+ct)$, $u_t = -cF'(x-ct)+cG'(x+ct)$ and $u_{tt} = c^2F''(x-ct) + c^2G''(x+ct)$ it follows that

$$u_{tt} - c^2 u_{xx} = c^2F''(x-ct) + c^2G''(x+ct) - c^2\left(F''(x-ct) + G''(x+ct)\right) = 0$$

for all x, t.

(b) [8p] Use the initial conditions to express f and g in terms of F and G.

$f(x) = u(x,0) = F(x) + G(x)$,
$g(x) = u_t(x,0) = -cF'(x) + cG'(x)$.

(c) [8p] Use your answer to part (b) to show that

$$\frac{1}{c}\int_0^x g(z)\ dz = -F(x) + G(x).$$

[HINT: You may assume that $F(0) = G(0)$]

Integrating the second equation from (b), we get

$$\begin{aligned} \frac{1}{c}\int_0^x g(z)\ dz &= \frac{1}{c}\int_0^x -cF'(z) + cG'(z)\ dz = \int_0^x -F'(z) + G'(z)\ dz \\ &= -F(x) + F(0) + G(x) - G(0) = -F(x) + G(x) \end{aligned}$$

by the hint.

(d) [9p] Show that

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\ dz.$$

By (b) and (c), we have

$$\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\ dz = \frac{1}{2}\Big(F(x) + G(x)\Big) - \frac{1}{2}\Big(-F(x) + G(x)\Big) = F(x).$$

(e) [9p] Show that

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\ dz.$$

Similarly

$$\frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(z)\ dz = \frac{1}{2}\Big(F(x) + G(x)\Big) + \frac{1}{2}\Big(-F(x) + G(x)\Big) = G(x).$$

(f) [8p] Finally, show that

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\ dz.$$

This is called *d'Alembert's solution* to (1).

By (a), (d) and (e), it follows that

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(z)\ dz + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(z)\ dz \\ &= \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\ dz. \end{aligned}$$

Question 2 (Characteristics)**.** Consider the PDE

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0 \tag{2}$$

with the initial condition

$$u(x,0) = \begin{cases} 1 & x < 2 \\ 3 & x > 2. \end{cases} \tag{3}$$

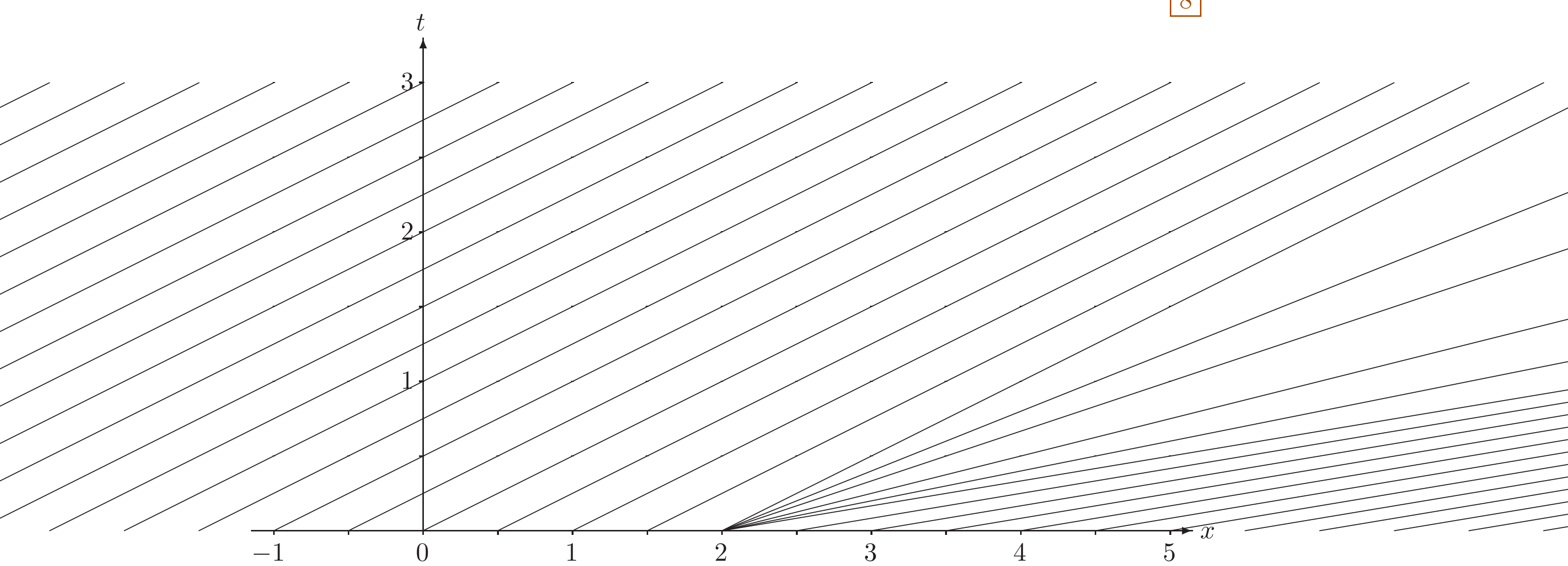
(a) [6p] Replace (2) by a system of 2 ODEs

$\frac{du}{dt} = 0$, $\frac{dx}{dt} = 2u$

(b) [12p] Plot the characteristics (t against x) for this problem.

The solution of $u' = 0$ is $u(x,t) = u(x(0),0)$, and the solution of $x' = 2u(x(0),0)$ is $x(t) = x(0) + 2u(x(0),0)t$. Thus $x(t) = x(0) + 2t$ or $x(t) = x(0) + 6t$ depending on $x(0)$. I expect to see fan-like characteristics eminating from $x(0) = 2$. $\boxed{4}$

$\boxed{8}$



(c) [2p] Does this problem have *fan-like characteristics*, *shock wave characteristics*, *neither* or *both*? [Mark ☑ only one box.]

☑ fan-like characteristics ☐ shock wave characteristics ☐ neither ☐ both

(d) [18p] Solve

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 1 & x < 2 \\ 3 & x > 2. \end{cases}$$

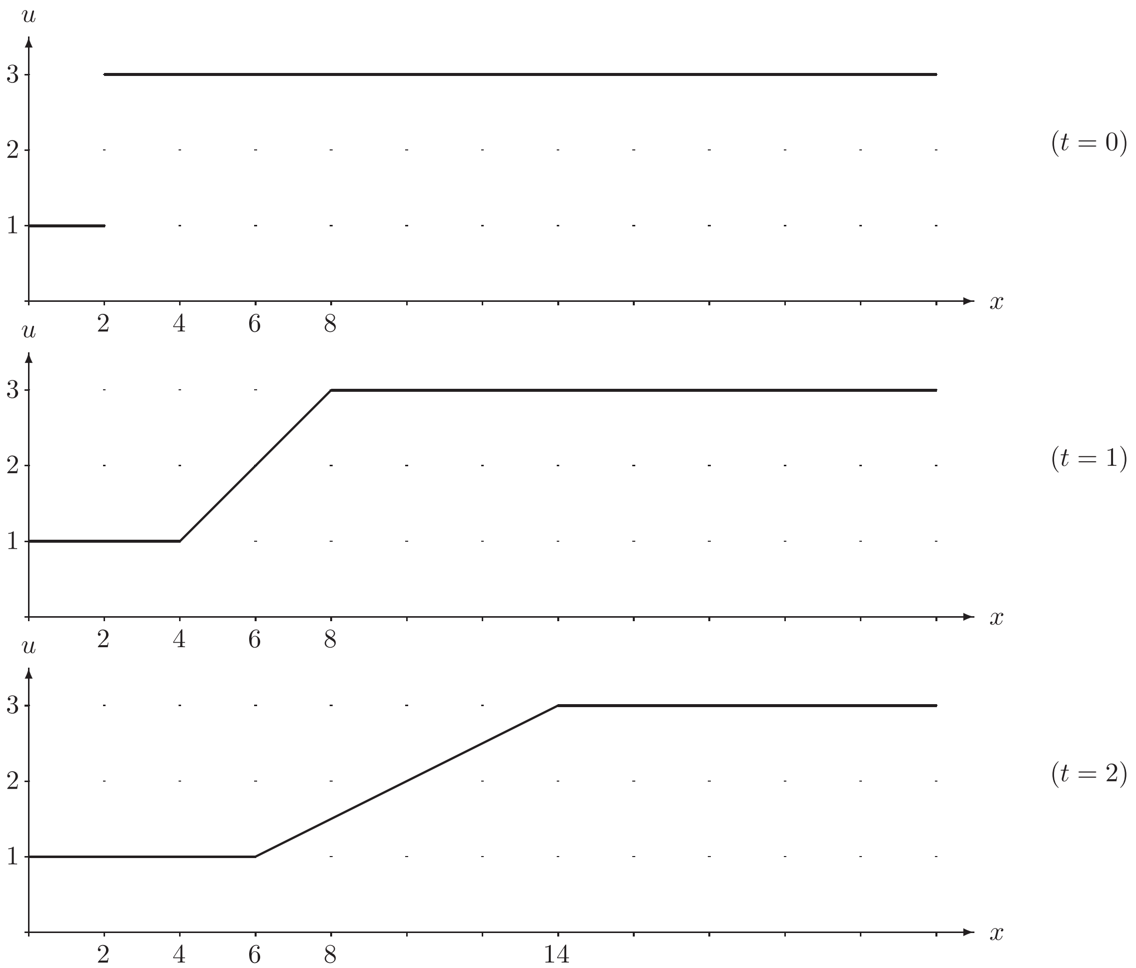
As above, $u(x,t) = u(x(0),0)$ and $x(t) = x(0) + 2u(x(0),0)t$. Therefore

$$u(x,t) = \begin{cases} 1 & x < 2+2t \\ ??? & 2+2t < x < 2+6t \\ 3 & x > 2+6t. \end{cases}$$

For the middle interval, we use the equation $x = x(0) + 2ut$ with $x(0) = 2$ to calculate that $u = \frac{x-2}{2t}$. Therefore

$$u(x,t) = \begin{cases} 1 & x < 2+2t \\ \frac{x-2}{2t} & 2+2t < x < 2+6t \\ 3 & x > 2+6t. \end{cases}$$

(e) [3×4p] Sketch the graph (u against x) of the solution at times $t = 0$, $t = 1$ and $t = 2$.



Question 3 (General Solution)**.** Consider the second order partial differential equation

$$\frac{1}{4}u_{xx} - \frac{1}{2}u_{xy} + \frac{1}{4}u_{yy} = y - x. \tag{4}$$

(a) [2p] Equation (4) is a

☐ hyperbolic PDE; ☑ parabolic PDE; ☐ elliptic PDE.

(b) [1p] Equation (4) is a

☐ homogeneous PDE; ☑ non-homogeneous PDE.

(c) [1p] Equation (4) is a

☑ linear PDE; ☐ quasilinear PDE; ☐ non-linear (and not quasilinear) PDE.

(d) [20p] Suppose that $\xi = y - x$ and $\eta = y + x$. Show that

$$\frac{1}{4}u_{xx} - \frac{1}{2}u_{xy} + \frac{1}{4}u_{yy} = u_{\xi\xi}.$$

Since $u_x = u_\xi \xi_x + u_\eta \eta_x = -u_\xi + u_\eta$ and $u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta$, it follows that
$u_{xx} = (u_x)_x = (-u_\xi + u_\eta)_x = -u_{\xi\xi}\xi_x - u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$,
$u_{xy} = (u_x)_y = (-u_\xi + u_\eta)_y = -u_{\xi\xi}\xi_y - u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y = -u_{\xi\xi} + u_{\eta\eta}$ and
$u_{yy} = (u_y)_y = (u_\xi + u_\eta)_y = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$.
Therefore

$$u_{xx} - 2u_{xy} + u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} + 2u_{\xi\xi} - 2u_{\eta\eta} + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = 4u_{\xi\xi}.$$

Dividing by 4 gives the required result. [Alternately, students may use the A^*, B^*, etc. formulae on page 2, if they explain what they are doing.]

(e) [20p] Find the general solution of $u_{\xi\xi} = \xi$.

Integrating the equation w.r.t. ξ gives

$$u_\xi = \int u_{\xi\xi}\, d\xi = \int \xi \, d\xi = \frac{1}{2}\xi^2 + F(\eta)$$

for some function F. Integrating w.r.t. ξ a second time gives

$$u(\xi, \eta) = \int u_\xi \, d\xi = \int \frac{1}{2}\xi^2 + F(\eta)\, d\xi = \frac{1}{6}\xi^3 + F(\eta)\xi + G(\eta)$$

for some function G.

(f) [6p] Now find the general solution of (4).

By parts (b) and (c), the general solution of the PDE is

$$u(x, y) = \frac{1}{6}(y - x)^3 + (y - x)F(y + x) + G(y + x).$$