2014.05.28 **MAT372 K.T.D.D. – Final Sınavın Çözümleri** N. Course

Soru 1 (Characteristics)**.** Consider the PDE

$$\frac{\partial u}{\partial t} - \frac{1}{3}u\frac{\partial u}{\partial x} = 0 \tag{1}$$

with the initial condition

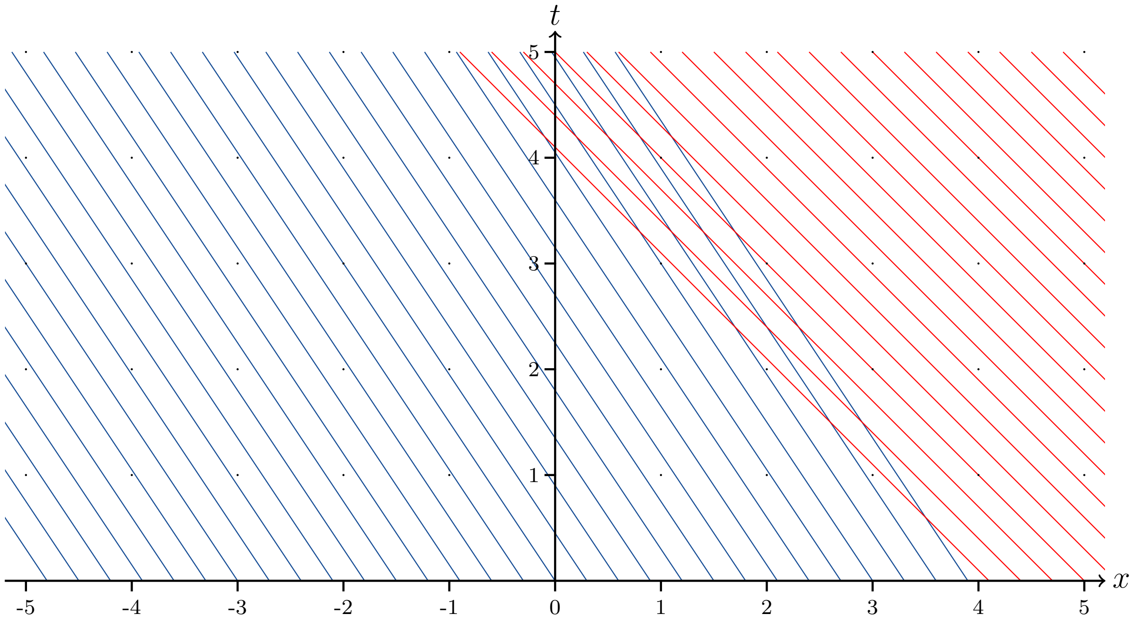
$$u(x,0) = \begin{cases} 2, & x < 4 \\ 3, & x > 4. \end{cases} \tag{2}$$

(a) [3p] Replace (1) by a system of 2 ODEs.

$\frac{du}{dt} = 0$, $\frac{dx}{dt} = -\frac{1}{3}u$

(b) [6p] Plot the characteristics (t against x) for this problem.

The solution of $u' = 0$ is $u(x,t) = u(x(0),0)$, and the solution of $x' = -\frac{1}{3}u(x(0),0)$ is $x(t) = x(0) - \frac{1}{3}u(x(0),0)t = \begin{cases} x(0) - \frac{2}{3}t & x(0) < 4 \\ x(0) - t & x(0) > 4 \end{cases}$. Thus...



(c) [1p] Does this problem have *fan-like characteristics*, *shock wave characteristics*, *neither* or *both*? [Mark ☑ only one box.]

☐ fan-like characteristics ☑ shock wave characteristics ☐ neither ☐ both

(d) [9p] Solve

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{1}{3}u\frac{\partial u}{\partial x} = 0 \\ u(x,0) = \begin{cases} 2, & x < 4 \\ 3, & x > 4. \end{cases} \end{cases}$$

As above, $u(x,t) = u(x(0),0)$ and

$$x(t) = x(0) - \tfrac{1}{3}u(x(0),0)t = \begin{cases} x(0) - \frac{2}{3}u(x(0),0) & x(0) < 4 \\ x(0) - u(x(0),0) & x(0) > 4. \end{cases}$$

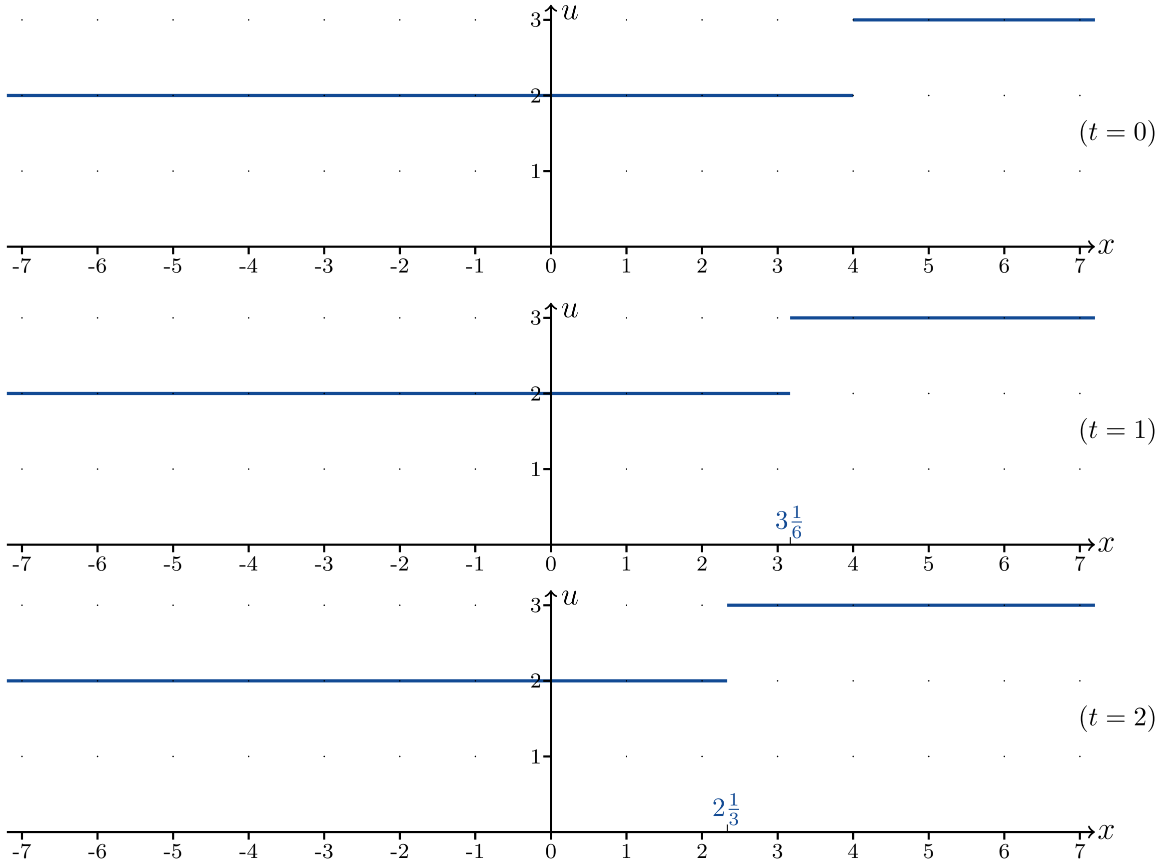
Since we have a shock wave problem, we need to calculate the shockwave characteristic $x_s(t)$. Define $q(u) = -\frac{1}{6}u^2$ (because $\frac{dq}{du} = -\frac{1}{3}u$). Then we have $[u] = 3 - 2 = 1$ and $[q] = -\frac{9}{6} - (-\frac{4}{6}) = -\frac{5}{6}$. Solving

$$\frac{dx_s}{dt} = \frac{[q]}{[u]} = -\frac{5}{6}$$

gives $x_s(t) = x_s(0) - \frac{5}{6}t = 4 - \frac{5}{6}t$. Therefore the solution is

$$u(x,t) = \begin{cases} 2 & x < 4 - \frac{5}{6}t \\ 3 & x > 4 - \frac{5}{6}t \end{cases}$$

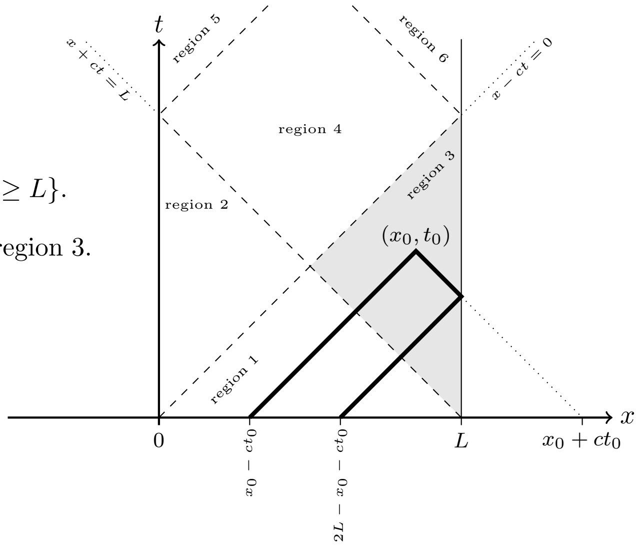
(e) [3 × 2p] Sketch the graph (u against x) of the solution at times $t = 0$, $t = 1$ and $t = 2$.



Soru 2 (Finite String Wave Equation)**.** Consider the wave equation on a string, of length L, with fixed ends:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \qquad t > 0 \\ u(x,0) = f(x) & f : (0,L) \to \mathbb{R} \\ u_t(x,0) = g(x) & g : (0,L) \to \mathbb{R} \\ u(0,t) = 0 & \\ u(L,t) = 0 & \end{cases} \tag{3}$$

where $c > 0$.



Let

region 3 $:= \{(x,t) : x \leq L, x - ct \geq 0 \text{ and } x + ct \geq L\}$.

In this question, you will calculate the solution in region 3.

(a) [5p] First show that

$$u(x,t) = F(x - ct) + G(x + ct)$$

solves the wave equation, $u_{tt} - c^2 u_{xx} = 0$, for any twice differentiable functions $F : (0,L) \to \mathbb{R}$ and $G : (0,L) \to \mathbb{R}$.

Since $u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$, $u_{tt}(x,t) = c^2F''(x-ct) + c^2G''(x+ct)$, $u_x(x,t) = F'(x-ct) + G'(x+ct)$ and $u_{xx}(x,t) = F''(x-ct) + G''(x+ct)$, we have that

$$u_{tt} - c^2 u_{xx} = (c^2F'' + c^2G'') - c^2(F'' + G'') = 0.$$

Using the initial conditions we can see that:

$$\begin{aligned} f(x) &= u(x,0) = F(x) + G(x) \\ g(x) &= u_t(x,0) = -cF'(x) + cG'(x) \end{aligned} \tag{4}$$

(b) [5p] Use (4) to show that

$$-F(x) + G(x) = \frac{1}{c} \int_0^x g(z)\, dz.$$

[HINT: You may assume that $F(0) = G(0)$]

$$\begin{aligned} \int_0^x g(z) dz &= \int_0^x -cF'(z) + cG'(z) dz = c\Big[-F(z) + G(z)\Big]_0^x \\ &= c\Big(-F(x) + G(x) + F(0) - G(0)\Big) = c(-F(x) + G(x)). \end{aligned}$$

Therefore

$$-F(x) + G(x) = \frac{1}{c} \int_0^x g(z)\, dz.$$

(c) [4p] Use (b) and (4) to show that

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(z)\, dz.$$

$$\frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(z)\ dz = \frac{1}{2}\big(F(x)+G(x)\big) - \frac{1}{2}\big(-F(x)+G(x)\big) = F(x).$$

(d) [4p] Next use (a) and (3) show that

$$G(L+ct) = -F(L-ct).$$

and that

$$G(z) = -F(2L-z) \qquad \text{for all } z \geq L.$$

$$0 = u(L,t) = F(L-ct) + G(L+ct) \implies G(L+ct) = -F(L-ct). \ \boxed{2}$$

For all $z \geq L$, let $t = \frac{1}{c}(z-L) \geq 0$. Then $L + ct = z$ and so

$$G(z) = G(L+ct) = -F(L-ct) = -F(L-(z-L)) = -F(2L-z). \ \boxed{2}$$

(e) [7p] Use (a), (c) and (d) to show that the solution in region 3 is

$$u(x,t) = \frac{f(x-ct) - f(2L-x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} g(\xi)\ d\xi + \frac{1}{2c}\int_0^{2L-x-ct} g(\xi)\ d\xi. \tag{5}$$

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= F(x-ct) - F(2L-(x+ct)) \\ &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x+ct} g(z)dz - \frac{1}{2}f(2L-x-ct) + \frac{1}{2c}\int_0^{2L-x-ct} g(z)dz \\ &= \frac{f(x-ct)-f(2L-x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} g(\xi)\ d\xi + \frac{1}{2c}\int_0^{2L-x-ct} g(\xi)\ d\xi. \end{aligned}$$

Soru 3 (Separation of Variables)**.** Consider the heat equation on a rod of length L:

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u_x(L,t) = 0. \end{cases} \tag{6}$$

(a) [5p] If $u(x,t) = X(x)T(t)$, show that X and T satisfy

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + k\lambda T = 0$$

for some constant $\lambda \in \mathbb{R}$.

Since $XT' = u_t = ku_{xx} = kX''T$, we have that $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$ $\boxed{2}$. The left-hand side is a function only of x; the right-hand side is a function only of t. Therefore both sides must be equal to a constant; equal to $-\lambda$ say $\boxed{2}$. Then $\frac{X''}{X} = -\lambda \implies X'' + \lambda X = 0$ and $\frac{T'}{kT} = -\lambda \implies T' + k\lambda T = 0$ $\boxed{1}$.

(b) [3p] What boundary conditions does X satisfy?

First note that $0 = u_x(0,t) = X'(0)T(t)$ and $0 = u_x(L,t) = X'(L)T(t)$. Since we don't want $T(t) = 0\ \forall t$, we must have... $\boxed{\text{optional}}$

$$\begin{cases} X'(0) = 0 \ \boxed{1.5} \\ X'(L) = 0. \ \boxed{1.5} \end{cases}$$

(c) [12p] By considering the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, find all the eigenvalues and eigenfunctions of

$$X'' + \lambda X = 0$$

subject to the boundary conditions that you wrote in part (b).

CASE 1: $\lambda < 0$.
The solution of $X'' + \lambda X = 0$ is $X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$. Then $0 = X'(0) = A\sqrt{-\lambda}e^0 - B\sqrt{-\lambda}e^0 \implies A = B$; and $0 = X'(L) = A\left(\sqrt{-\lambda}e^{\sqrt{-\lambda}L} - \sqrt{-\lambda}e^{-\sqrt{-\lambda}L}\right) \implies A = 0 \implies B = 0$. There are no eigenvalues and no non-trivial eigenfunctions in this case. $\boxed{3}$

CASE 2: $\lambda = 0$.
The solution of $X'' = 0$ is $X(x) = Ax + B$. Then $0 = X'(0) = A$ and $0 = X'(L) = A \implies A = 0$. We can choose any B. Therefore $\lambda_0 = 0$ is an eigenvalue and $X_0(x) = 1$ is an eigenfunction. $\boxed{3}$

CASE 3: $\lambda > 0$.
The solution of $X'' + \lambda X = 0$ is $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. So $0 = X'(0) = -A\sqrt{\lambda}\sin\sqrt{\lambda}0 + B\sqrt{\lambda}\cos\sqrt{\lambda}0 = B\sqrt{\lambda} \implies B = 0$; and $0 = X'(L) = -A\sqrt{\lambda}\sin\sqrt{\lambda}L$. Since we don't want $A = 0$, we must have that $\sin\sqrt{\lambda}L = 0$. So $\sqrt{\lambda}L = n\pi$, $n = 1, 2, 3, \ldots$. So $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ are eigenvalues, with eigenfunctions $X_n(x) = \cos\frac{n\pi x}{L}$. $\boxed{6}$

(d) [5p] Find the general solution of

$$\begin{cases} u_t = ku_{xx} & 0 < x < L, \quad 0 < t \\ u_x(0,t) = 0 \\ u_x(L,t) = 0. \end{cases}$$

The solution of $T_n' + k\lambda_n T_n = 0$ is $T_n(t) = a_n e^{-k\lambda_n t}$ $\boxed{2}$. So the general solution of the problem is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L} \quad \boxed{3}$$

for some constants a_n.

Soru 4 (Fourier Transforms). [25p] Use the Fourier Transform to solve

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 13\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 < t < \infty, \\ u(x,0) = \frac{1}{1+x^2} \\ u_t(x,0) = 0. \end{cases} \tag{7}$$

[HINT: You may give your answer as a double integral, or as a convolution of 2 functions.]

Taking Fourier Transforms, the problem becomes

$$\begin{cases} U_{tt} = 13(i\omega)^4 U - (i\omega)^2 U = (13\omega^4 + \omega^2)U \\ U(\omega, 0) = \frac{1}{2}e^{-|\omega|} \\ U_t(\omega, 0) = 0. \qquad \boxed{5} \end{cases}$$

The general solution of $U_{tt} = (13\omega^4 + \omega^2)U$ is

$$U(\omega, t) = A(\omega)\cosh\left[\left(\sqrt{13\omega^4 + \omega^2}\right)t\right] + B(\omega)\sinh\left[\left(\sqrt{13\omega^4 + \omega^2}\right)t\right].$$

Using the initial conditions, we obtain $B = 0$ and $A(\omega) = \frac{1}{2}e^{-|\omega|}$. Therefore

$$U(\omega, t) = \frac{1}{2}e^{-|\omega|}\cosh\left[\left(\sqrt{13\omega^4 + \omega^2}\right)t\right]. \qquad \boxed{10}$$

If we define $G(\omega, t) = \cosh\left[\left(\sqrt{13\omega^4 + \omega^2}\right)t\right]$, then we have $U(\omega, t) = \frac{1}{2}e^{-|\omega|}G(\omega, t)$. So $u(x, t) = g(x, t) * \frac{1}{1+x^2}$ where

$$g(x, t) = \mathcal{F}^{-1}[G](x, t) = \int_{-\infty}^{\infty} \cosh\left[\left(\sqrt{13\omega^4 + \omega^2}\right)t\right] e^{i\omega x} d\omega. \qquad \boxed{5}$$

Therefore the solution to (7) is

$$u(x, t) = g(x, t) * \frac{1}{1 + x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{g(x - \xi, t)}{1 + \xi^2}\, d\xi. \qquad \boxed{5}$$

Soru 5 (Fourier Cosine Series)**.** Define the function $f : [0, \pi] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1 & x = 0,\ x = \frac{\pi}{2},\ \text{or } x = \pi \\ 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi. \end{cases} \tag{8}$$

(a) [6p] Show that

$$\{\cos nx : n \in \mathbb{N}\}$$

is an orthogonal system on $[-\pi, \pi]$.

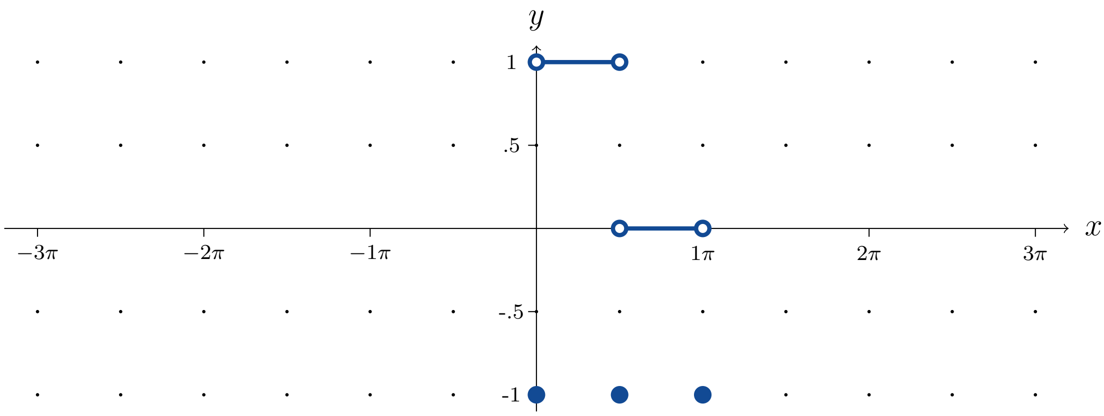
[HINT: $\cos(A + B) = \cos A\cos B - \sin A\sin B$, so $\cos(A + B) + \cos(A - B) = ?$ and $\cos(A + B) - \cos(A - B) = ?$]

Let $n \neq m$, $n, m \in \mathbb{N}$. Then

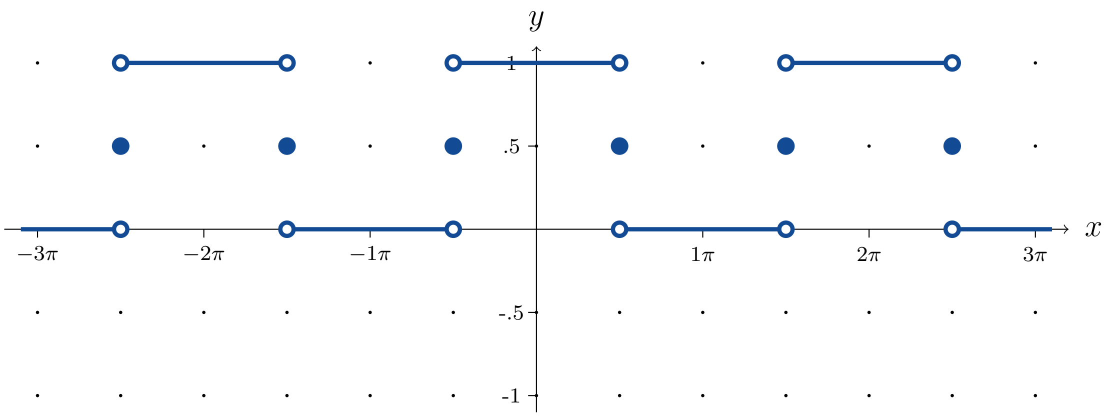
$$\begin{aligned} \langle\cos nx, \cos mx\rangle &= \int_{-\pi}^{\pi} \cos nx\cos mx\ dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n + m)x + \cos(n - m)x\ dx \\ &= \frac{1}{2}\left[\frac{1}{n + m}\sin(n + m)x + \frac{1}{n - m}\sin(n - m)x\right]_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

Therefore $\{\cos nx : n \in \mathbb{N}\}$ is an orthogonal system on $[-\pi, \pi]$.

(b) [2p] Sketch f.



(c) [7p] Sketch the Fourier **Cosine** Series of f.



$$f(x) = \begin{cases} -1 & x = 0,\ x = \frac{\pi}{2},\ \text{or } x = \pi \\ 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi. \end{cases}$$

(d) [10p] Calculate the coefficients (a_k, $k = 0, 1, 2, 3, \ldots$) of the Fourier **Cosine** Series of f.

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx \\ &= \frac{2}{\pi}\int_0^{\pi} f(x)\ dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1\ dx + \int_{\frac{\pi}{2}}^{\pi} 0\ dx \\ &= \frac{2}{\pi}\frac{\pi}{2} = 1 \qquad \boxed{3} \end{aligned}$$

and

$$\begin{aligned} a_k &= \ldots = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos kx\ dx \\ &= \frac{2}{\pi}\left[\frac{1}{k}\sin kx\right]_0^{\frac{\pi}{2}} = \frac{2}{k\pi}\sin\frac{k\pi}{2} \\ &= \begin{cases} \frac{2}{k\pi} & k = 1, 5, 9, \ldots \\ 0 & k = 2, 4, 6, 8, \ldots \\ -\frac{2}{k\pi} & k = 3, 7, 11, \ldots \end{cases} \qquad \boxed{7} \end{aligned}$$

Therefore $\boxed{\text{optional}}$

$$f(x) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{k\pi}\sin\frac{k\pi}{2}\cos kx$$