2014.04.09 **MAT372 K.T.D.D. – Ara Sınavın Çözümleri** N. Course

Soru 1 (Canonical Forms)**.** Consider the partial differential equation

$$xu_{xx} + u_{yy} = x^2 \tag{1}$$

(where u_x means $\frac{\partial u}{\partial x}$ etc).

(a) [1p] Equation (1) is a

☐ 1st order PDE; ☑ 2nd order PDE; ☐ 3rd order PDE; ☐ 4th order PDE.

(b) [1p] Equation (1) is a

☐ homogeneous PDE; ☑ non-homogeneous PDE.

(c) [1p] Equation (1) is a

☑ linear PDE; ☐ quasilinear PDE; ☐ non-linear (and not quasilinear) PDE.

(d) [6p] For each $(x, y) \in \mathbb{R}^2$, classify (1) as hyperbolic, parabolic or elliptic.

Since $\Delta = B^2 - 4AC = 0^2 - 4 \times x \times 1 = -4x$, the PDE is hyperbolic for $x < 0$, parabolic for $x = 0$ and elliptic for $x > 0$.

For parts (e), (f) and (g), suppose that $x > 0$.

(e) [10p] Find the characteristic equation of (1).

The characteristic equation of (1) is

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{\pm\sqrt{-4x}}{2x} = \pm\frac{i}{\sqrt{x}}.$$

$$xu_{xx} + u_{yy} = x^2 \tag{1}$$

(f) [10p] Find the characteristic curves of (1).

Solving $\frac{dy}{dx} = \pm i x^{-\frac{1}{2}}$, we see that $y = \pm 2ix^{\frac{1}{2}} + c = \pm 2i\sqrt{x} + c$

(g) [21p] Find a canonical form for (1).
[HINT: I don't want to see x or y in your final answer.]

By part (f), we choose $\xi = y - 2i\sqrt{x}$ and $\eta = y + 2i\sqrt{x}$. Then, because we are studying an elliptic PDE, we choose $\alpha = \mathrm{Re}(\xi) = y$ and $\beta = \mathrm{Im}(\xi) = -2\sqrt{x}$. We can calculate

$$\alpha_x = 0,\ \alpha_y = 1,\ \alpha_{xx} = 0,\ \alpha_{xy} = 0,\ \alpha_{yy} = 0,$$

and

$$\beta_x = -x^{-\frac{1}{2}},\ \beta_y = 0,\ \beta_{xx} = \frac{1}{2}x^{-\frac{3}{2}},\ \beta_{xy} = 0,\ \beta_{yy} = 0.$$

Therefore

$$\begin{aligned}
A^{**} &= 0 + 0 + 1 = 1 \\
B^{**} &= 0 + 0 + 0 = 0 \\
C^{**} &= x(-x^{-\frac{1}{2}})^2 + 0 + 0 = 1 \\
D^{**} &= 0 + 0 + 0 + 0 + 0 = 0 \\
E^{**} &= x\left(\frac{1}{2}x^{-\frac{3}{2}}\right) + 0 + 0 + 0 + 0 = \frac{1}{2}x^{-\frac{1}{2}} \\
F^{**} &= 0 \\
G^{**} &= x^2.
\end{aligned}$$

Hence

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{2}x^{-\frac{1}{2}}u_\beta = x^2.$$

The final step is to substitute for x,

$$u_{\alpha\alpha} + u_{\beta\beta} - \frac{1}{\beta}u_\beta = \left(-\frac{1}{2}\beta\right)^4 = \frac{\beta^4}{16}.$$

Therefore, the canonical form for (1) is

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_\beta + \frac{\beta^4}{16}.$$

Soru 2 (The method of characteristics)**.** Consider

$$\begin{cases} \frac{\partial u}{\partial t} - 3t^2\frac{\partial u}{\partial x} = -u \\ u(x,0) = 2e^x. \end{cases} \tag{2}$$

(a) [40p] Use the method of characteristics to solve (2).

I think that this is the easiest question on this exam – I hope that you choose this one.

First we replace the PDE by a system of 2 ODEs:

$$\frac{du}{dt} = -u \qquad \text{and} \qquad \frac{dx}{dt} = -t^2.$$

The solution of the second ODE is

$$x(t) = -t^3 + x(0)$$

and the solution of the first ODE is

$$u(x(t),t) = Ke^{-t} = u(x(0),t)e^{-t} = 2e^{x(0)}e^{-t} = 2e^{x(0)-t}.$$

The final step is to substitute for $x(0)$. Therefore the solution of (2) is

$$u(x,t) = 2e^{x(0)-t} = 2e^{x+t^3-t}.$$

$$\begin{cases} \frac{\partial u}{\partial t} - 3t^2\frac{\partial u}{\partial x} = -u \\ u(x,0) = 2e^x. \end{cases} \tag{2}$$

(b) [10p] To check your answer to part (a), calculate u_t, u_x, $(u_t - 3t^2 u_x)$ and $u(x, 0)$.

Since
$$u(x,t) = 2e^{x+t^3-t}$$
we have that
$$u_t = 2(3t^2-1)e^{x+t^3-t}$$
$$u_x = 2e^{x+t^3-t}.$$
Therefore
$$u_t - 3t^2 u_x = (2(3t^2-1)e^{x+t^3-t}) - 3t^2(2e^{x+t^3-t}) = -2e^{x+t^3-t} = -u$$
so the PDE is satisfied.
Finally, clearly
$$u(x,0) = 2e^{x+0^3-0} = 2e^x$$
so the initial condition is also satisfied.

Soru 3 (General Solution). Suppose that $\xi = y - 9x$ and $\eta = y - x$.

(a) [20p] **Use the chain rule** (e.g. $u_x = u_\xi \xi_x + u_\eta \eta_x$, etc.) to show that
$$u_{xx} + 10u_{xy} + 9u_{yy} = -64u_{\xi\eta}.$$

Since
$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = -9u_\xi - u_\eta \\
u_y &= u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta \\
u_{xx} &= (u_x)_\xi \xi_x + (u_x)_\eta \eta_x = (-9u_{\xi\xi} - u_{\eta\xi})(-9) + (-9u_{\xi\eta} - u_{\eta\eta})(-1) = 81u_{\xi\xi} + 18u_{\xi\eta} + u_{\eta\eta} \\
u_{xy} &= (u_x)_\xi \xi_y + (u_x)_\eta \eta_y = (-9u_{\xi\xi} - u_{\eta\xi})(1) + (-9u_{\xi\eta} - u_{\eta\eta})(1) = -9u_{\xi\xi} - 10u_{\xi\eta} - u_{\eta\eta} \\
u_{yy} &= (u_y)_\xi \xi_y + (u_y)_\eta \eta_y = (u_{\xi\xi} + u_{\eta\xi})(1) + (u_{\xi\eta} + u_{\eta\eta})(1) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}
\end{aligned}$$
it follows that
$$\begin{aligned}
u_{xx} + 10u_{xy} + 9u_{yy} &= \big(81u_{\xi\xi} + 18u_{\xi\eta} + u_{\eta\eta}\big) + 10\big(-9u_{\xi\xi} - 10u_{\xi\eta} - u_{\eta\eta}\big) \\
&\quad + 9\big(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\big) \\
&= u_{\xi\xi}\big(81 - 90 + 9\big) + u_{\xi\eta}\big(18 - 100 + 18\big) + u_{\eta\eta}\big(1 - 10 + 9\big) \\
&= -64u_{\xi\eta}.
\end{aligned}$$

(b) [5p] Show that
$$y = \frac{9\eta - \xi}{8}.$$

Since $\xi = y - 9x$ and $\eta = y - x$, it follows that
$$9\eta - \xi = 9(y - x) - (y - 9x) = 8y.$$
Therefore
$$y = \frac{9\eta - \xi}{8}.$$

Now consider the second order partial differential equation
$$u_{xx} + 10u_{xy} + 9u_{yy} = y \tag{3}$$
(where u_x means $\frac{\partial u}{\partial x}$ etc).

(c) [5p] Equation (3) is a

☑ hyperbolic PDE; ☐ parabolic PDE; ☐ elliptic PDE.

(d) [20p] Find the general solution of (3).

[HINT: I don't want to see ξ or η in your final answer.]

By parts (a) and (b),

$$-64u_{\xi\eta} = u_{xx} + 10u_{xy} + 9u_{yy} = y = \frac{9\eta - \xi}{8}.$$

So

$$512u_{\xi\eta} = \xi - 9\eta.$$

Integrating (wrt η) gives

$$512u_{\xi} = \int 512u_{\xi\eta}\; d\eta = \int \xi - 9\eta\; d\eta = \xi\eta - \frac{9}{2}\eta^2 + f(\xi)$$

for some function f. Then integrating (wrt ξ) gives

$$512u(\xi,\eta) = \int 512u_{\xi}\; d\xi = \int \xi\eta - \frac{9}{2}\eta^2 + f(\xi)\; d\xi = \frac{1}{2}\xi^2\eta - \frac{9}{2}\eta^2\xi + 512F(\xi) + 512G(\eta)$$

for some functions F and G.

The final step is to change back to x and y: The general solution of (3) is

$$u(x,y) = \frac{1}{1024}(y-9x)^2(y-x) - \frac{9}{1024}(y-x)^2(y-9x) + F(y-9x) + G(y-x).$$