2015.05.20 **MAT372 K.T.D.D. – Final Sınavın Çözümleri** N. Course

Soru 1 (Separation of Variables).

[25p] Explain the method of *Separation of Variables* for partial differential equations.

[25p] *Değişkenleri Ayırma Yöntemi*ni kısmi türevli diferansiyel denklemleri için açıklayınız.

Imagine that you are explaining the method of *Separation of Variables* to someone who hasn't studied this course. How would you explain it? This question should take you ≈ 25 minutes.

You might like to include:

- the main concepts of this method;
- an explaination of the *separation constant*
- an explaination of *eigenvalues* and *eigenfunctions*;
- an example of your choosing.

Bu dersi almamış birisine *Değişkenleri Ayırma Yöntemi*ni anlatmanız gerektiğini varsayalım. Bu yöntemi nasıl anlatırdınız? Bu soruyu cevaplamak yaklaşık 25 dakikanızı alacaktır.

Bu soruyu cevaplarken aşağıdaki noktalara da yer veriniz:

- bu yöntemin temel kavramları;
- *ayırma sabitinin* açıklaması;
- *özdeğer* ve *özişlev*'in açıklamaları;
- sizin seçeğiniz bir örnek.

The are many possible solutions to this question.

Marks will be given generously.

Soru 2 (Method of Characteristics). Consider

$$\frac{\partial u}{\partial t} + t^2 u \frac{\partial u}{\partial x} = 5. \tag{1}$$

(a) [1p] Equation (1) is

☐ linear; ☑ non-linear AND quasilinear; ☐ non-linear, but not quasilinear;

(b) [17p] Use the method of characteristics to solve

$$\begin{cases} \frac{\partial u}{\partial t} + t^2 u \frac{\partial u}{\partial x} = 5 \\ u(x,0) = x. \end{cases} \tag{2}$$

Using the Method of Characteristics, we can change (1) into two ODEs:

$$\frac{du}{dt} = 5 \qquad\qquad \frac{dx}{dt} = t^2 u.$$

Since the latter equation includes u, we start with the former.

The solution to $u' = 5$ is $u(x(t), t) = 5t + K$. At $t = 0$, we have $x_0 = u(x_0, 0) = K$. Therefore

$$u(x(t), t) = 5t + x_0.$$

The latter ODE then becomes $x' = 5t^3 + x_0 t^2$ which has solution

$$x(t) = \frac{5}{4}t^4 + \frac{1}{3}x_0 t^3 + x_0 = \frac{5}{4}t^4 + \left(\frac{1}{3}t^3 + 1\right) x_0.$$

Solving for x_0, we have that

$$x_0 = \frac{x - \frac{5}{4}t^4}{\frac{t^3}{3} + 1}.$$

Therefore

$$u(x, t) = 5t + \frac{x - \frac{5}{4}t^4}{\frac{t^3}{3} + 1}$$

is the solution to (2).

$$\frac{\partial u}{\partial t} + t^2 u \frac{\partial u}{\partial x} = 5 \tag{1}$$

(c) [7p] Check your answer to part (b) by differentiating your solution $u(x, t)$ and calculating $(u_t + t^2 u u_x)$.

My answer to part (b) is

$$u(x, t) = 5t + \frac{x - \frac{5}{4}t^4}{\frac{t^3}{3} + 1}.$$

We can calculate that

$$u_t = 5 + \frac{(-5t^3)\left(\frac{t^3}{3} + 1\right) - \left(x - \frac{5}{4}t^4\right)(t^2)}{\left(\frac{t^3}{3} + 1\right)^2} = 5 + \frac{\frac{5}{4}t^6 - \frac{5}{3}t^6 - 5t^3 - xt^2}{\left(\frac{t^3}{3} + 1\right)^2}$$

and

$$u_x = \frac{1}{\frac{t^3}{3} + 1}.$$

Therefore

$$\begin{aligned}
u_t + t^2 u u_x &= 5 + \frac{\frac{5}{4}t^6 - \frac{5}{3}t^6 - 5t^3 - xt^2}{\left(\frac{t^3}{3} + 1\right)^2} + t^2 \left(5t + \frac{x - \frac{5}{4}t^4}{\frac{t^3}{3} + 1}\right)\left(\frac{1}{\frac{t^3}{3} + 1}\right) \\
&= 5 + \frac{\frac{5}{4}t^6 - \frac{5}{3}t^6 - 5t^3 - xt^2}{\left(\frac{t^3}{3} + 1\right)^2} + t^2 \left(\frac{\frac{5}{3}t^4 + 5t + x - \frac{5}{4}t^4}{\frac{t^3}{3} + 1}\right)\left(\frac{1}{\frac{t^3}{3} + 1}\right) \\
&= 5 + \frac{\frac{5}{4}t^6 - \frac{5}{3}t^6 - 5t^3 - xt^2}{\left(\frac{t^3}{3} + 1\right)^2} + \frac{\frac{5}{3}t^6 + 5t^3 + xt^2 - \frac{5}{4}t^6}{\left(\frac{t^3}{3} + 1\right)^2} \\
&= 5
\end{aligned}$$

as expected.

Soru 3 (The Parallelogram Rule). Consider the wave equation on a string, of length L, with fixed ends:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L \quad t > 0 \\ u(x,0) = f(x) & f : (0,L) \to \mathbb{R} \\ u_t(x,0) = g(x) & g : (0,L) \to \mathbb{R} \\ u(0,t) = 0 & \\ u(L,t) = 0 & \end{cases} \tag{3}$$

where $c > 0$.

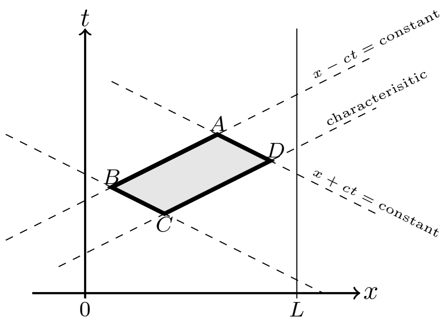
(a) [5p] First show that

$$u(x,t) = F(x-ct) + G(x+ct)$$

solves the wave equation, $u_{tt} - c^2 u_{xx} = 0$, for any twice differentiable functions $F : (0,L) \to \mathbb{R}$ and $G : (0,L) \to \mathbb{R}$.

Since $u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$, $u_{tt}(x,t) = c^2F''(x-ct) + c^2G''(x+ct)$, $u_x(x,t) = F'(x-ct) + G'(x+ct)$ and $u_{xx}(x,t) = F''(x-ct) + G''(x+ct)$, we have that

$$u_{tt} - c^2 u_{xx} = (c^2F'' + c^2G'') - c^2(F'' + G'') = 0.$$



$A = (x_1, t_1)$
$B = (x_2, t_2)$
$C = (x_3, t_3)$
$D = (x_4, t_4)$

Suppose that

- the parallelogram $ABCD$ is contained in $[0,L] \times [0,\infty)$;
- each of the edges of the parallelogram lies on characteristics of the wave equation; and
- $u(x,t) = F(x-ct) + G(x+ct)$.

(b) [20p] Prove that

$$u(A) + u(C) = u(B) + u(D).$$

Let F and G be as in part (a). Abusing notation; when I write $F(A)$, I mean $F(x_1 - ct_1)$. Similarly $G(C) := G(x_3 + ct_3)$, etc.

Now, since A and B lie on the same characteristic $x - ct =$ constant, we must have that

$$F(A) = F(x_1 - ct_1) = F(\text{constant}) = F(x_2 - ct_2) = F(B).$$

Similarly $F(C) = F(D)$, $G(A) = G(D)$ and $G(B) = G(C)$.

It is then straightforward to see that

$$\begin{aligned} u(A) + u(C) &= F(A) + G(A) + F(C) + G(C) \\ &= F(B) + G(D) + F(D) + G(B) \\ &= u(B) + u(D). \end{aligned}$$

Soru 4 (Fourier Transforms). Let $\mathcal{F}$ denote the Fourier Transform operator with respect to x.

(a) [7p] Suppose that $v : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Show that

$$\mathcal{F}\left[\frac{\partial v}{\partial t}\right](\omega, t) = \frac{\partial}{\partial t}\mathcal{F}[v](\omega, t)$$

for all $\omega, t \in \mathbb{R}$.

$$\mathcal{F}\left[\frac{\partial v}{\partial t}\right](\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial v}{\partial t}(x,t)e^{-i\omega x}dx = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} v(x,t)e^{-i\omega x}dx\right) = \frac{\partial}{\partial t}\mathcal{F}[v](\omega, t).$$

(b) [18p] Use the Fourier Transform to solve

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + u\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial x}\right)^2 = 0, & -\infty < x < \infty, \quad 0 < t < \infty, \\ u(x,0) = f(x) \\ u_t(x,0) = 0. \end{cases} \tag{4}$$

Taking Fourier Transforms, the problem becomes

$$\begin{cases} 0 = U_{tt} + U(-\omega^2 U) - (i\omega U)^2 = U_{tt} \\ U(\omega, 0) = F(\omega) \\ U_t(\omega, 0) = 0. \qquad \boxed{5} \end{cases}$$

The general solution of $U_{tt} = 0$ is

$$U(\omega, t) = A(\omega)t + B(\omega).$$

Using the initial conditions, we obtain $A = 0$ and $B(\omega) = F(\omega)$. Therefore

$$U(\omega, t) = F(\omega). \qquad \boxed{10}$$

Therefore the solution to (4) is

$$u(x,t) = f(x,t). \qquad \boxed{3}$$

Soru 5 (Fourier Sine Series). Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{2} & x = 0, \ x = 1, \\ -\frac{1}{2} & 0 < x \le \frac{1}{2} \\ x & \frac{1}{2} < x < 1. \end{cases} \tag{5}$$

(a) [7p] Show that

$$\{\sin n\pi x : n \in \mathbb{N}\}$$

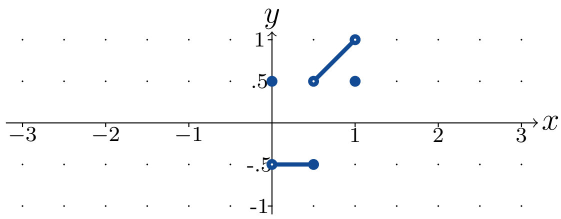
is an orthogonal system on $[-1, 1]$.
[HINT: $\cos(A+B) = \cos A\cos B - \sin A \sin B$, so $\cos(A+B) + \cos(A-B) =?$ and $\cos(A+B) - \cos(A-B) =?$]

Let $n \neq m$, $n, m \in \mathbb{N}$. Then

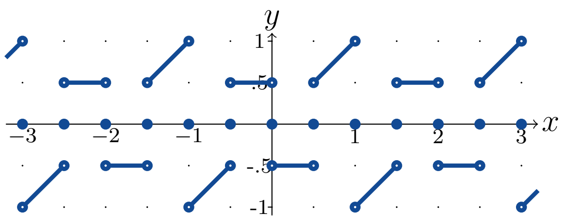
$$\begin{aligned}
\langle \sin nx, \sin mx \rangle &= \int_{-1}^{1} \sin nx \sin mx \; dx \\
&= \frac{1}{2} \int_{-1}^{1} -\cos(n+m)\pi x + \cos(n-m)\pi x \; dx \\
&= \frac{1}{2} \left[\frac{-1}{(n+m)\pi} \sin(n+m)\pi x + \frac{1}{(n-m)\pi} \sin(n-m)\pi x \right]_{-1}^{1} \\
&= 0
\end{aligned}$$

Therefore $\{\sin n\pi x : n \in \mathbb{N}\}$ is an orthogonal system on $[-1, 1]$.

(b) [1p] Sketch f.



(c) [7p] Sketch the Fourier **Sine** Series of f.



$$f(x) = \begin{cases} \frac{1}{2} & x = 0, \; x = 1, \\ -\frac{1}{2} & 0 < x \leq \frac{1}{2} \\ x & \frac{1}{2} < x < 1. \end{cases}$$

(d) [10p] Calculate the coefficients (b_k, $k = 1, 2, 3, \ldots$) of the Fourier **Sine** Series of f.

$$\begin{aligned}
b_k &= \frac{1}{L} \int_{-L}^{L} f(x) \sin k\pi x \; dx = \frac{2}{L} \int_{0}^{L} f(x) \sin k\pi x \; dx \\
&= 2 \int_{0}^{\frac{1}{2}} \left(-\frac{1}{2}\right) \sin k\pi x \; dx + 2 \int_{\frac{1}{2}}^{1} x \sin k\pi x \; dx \\
&= \int_{0}^{\frac{1}{2}} -\sin k\pi x \; dx + 2 \left[x \frac{-\cos k\pi x}{k\pi} \right]_{\frac{1}{2}}^{1} - 2 \int_{\frac{1}{2}}^{1} \frac{-\cos k\pi x}{k\pi} dx \\
&= \frac{1}{k\pi} \left[\cos k\pi x \right]_{0}^{\frac{1}{2}} - \frac{2}{k\pi} \left[x \cos k\pi x \right]_{\frac{1}{2}}^{1} + \frac{2}{k^2\pi^2} \left[\sin k\pi x \right]_{\frac{1}{2}}^{1} \\
&= \frac{1}{k\pi} \left(\cos \frac{1}{2} k\pi - 1 \right) - \frac{2}{k\pi} \left((-1)^k - \frac{1}{2} \cos \frac{1}{2} k\pi \right) + \frac{2}{k^2\pi^2} \left(0 - \sin \frac{1}{2} k\pi \right) \\
&= \frac{1}{k\pi} \left(2 \cos \frac{1}{2} k\pi - 1 - 2(-1)^k \right) - \frac{2 \sin \frac{1}{2} k\pi}{k^2\pi^2} \\
&= \begin{cases} \frac{1}{k\pi} - \frac{2}{k^2\pi^2} & k = 1, 5, 9, \ldots \\ -\frac{5}{k\pi} & k = 2, 6, 10, \ldots \\ \frac{1}{k\pi} + \frac{2}{k^2\pi^2} & k = 3, 7, 11, \ldots \\ -\frac{1}{k\pi} & k = 4, 8, 12, \ldots \end{cases}
\end{aligned}$$