OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2015.03.25 **MAT372 K.T.D.D. – Ara Sınavın Çözümleri** N. Course

Soru 1 (Characteristics)**.** Consider the PDE

$$\frac{\partial u}{\partial t} - \frac{u}{4}\frac{\partial u}{\partial x} = 0 \tag{1}$$

with the initial condition

$$u(x,0) = \begin{cases} 4 & x < 4 \\ 3 & x > 4. \end{cases} \tag{2}$$

(a) [6p] Replace (1) by a system of 2 ODEs

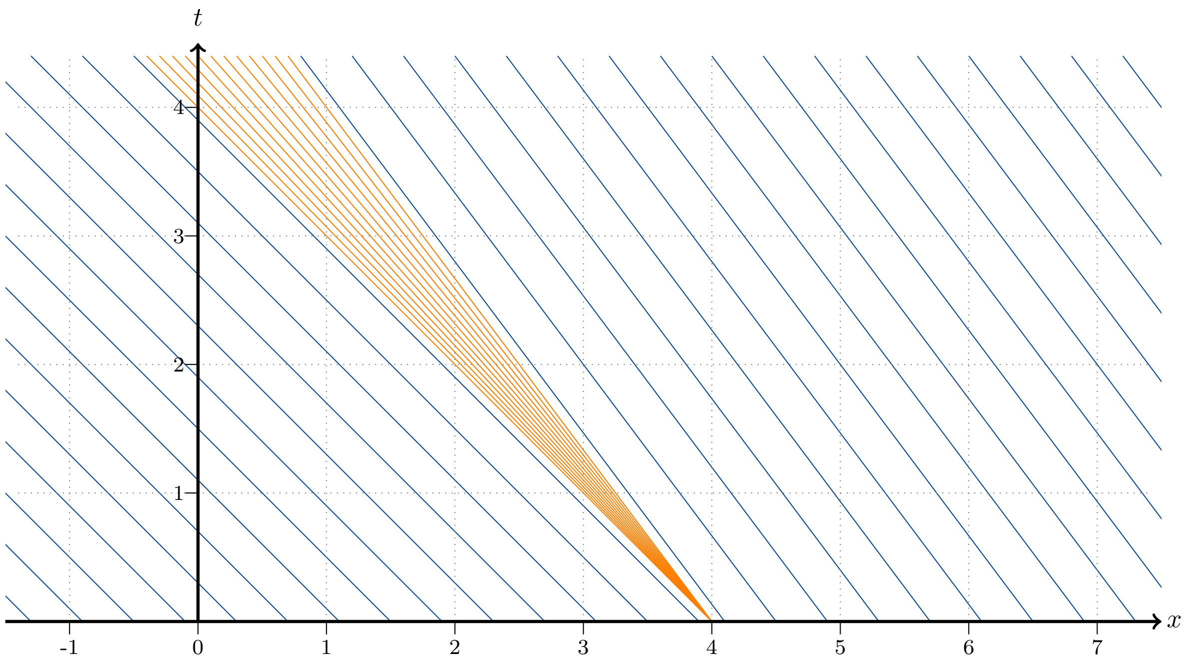
$\frac{du}{dt} = 0$, $\frac{dx}{dt} = -\frac{u}{4}$

(b) [12p] Plot the characteristics (t against x) for this problem.

The solution of $u' = 0$ is $u(x,t) = u(x(0),0)$, and the solution of $x' = -\frac{1}{4}u(x(0),0)$ is

$$x(t) = x(0) - \frac{1}{4}u(x(0),0)t = \begin{cases} x(0) - t & x(0) < 4 \\ x(0) - \frac{3}{4}t & x(0) > 4. \end{cases}$$

Thus...



(c) [2p] Does this problem have *fan-like characteristics*, *shock wave characteristics*, *neither* or *both*? [Mark ☑ only one box.]

☑ fan-like characteristics ☐ shock wave characteristics ☐ neither ☐ both

(d) [18p] Solve

$$\frac{\partial u}{\partial t} - \frac{u}{4}\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 4 & x < 4 \\ 3 & x > 4. \end{cases}$$

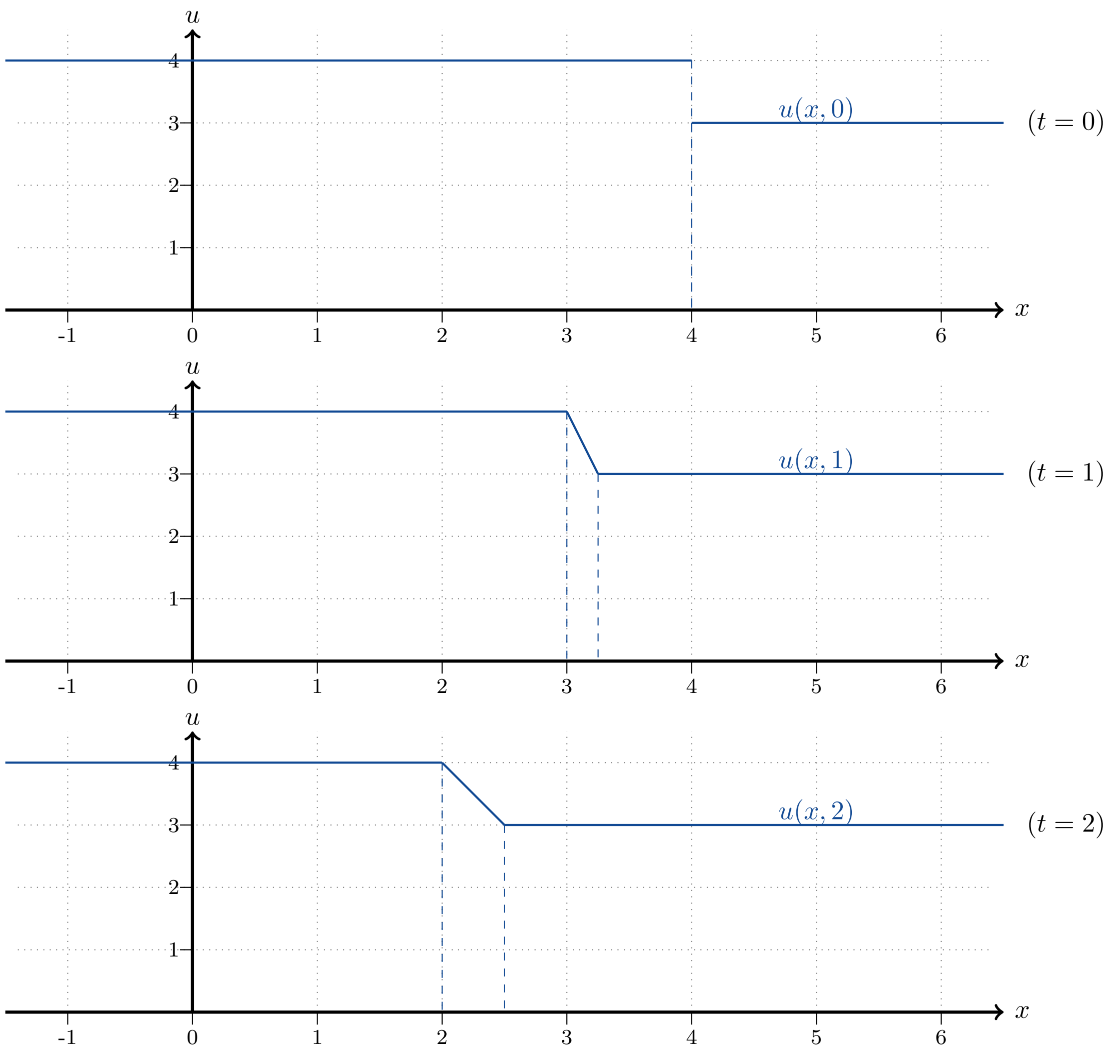
As above, $u(x,t) = u(x(0),0)$ and $x(t) = x(0) - \frac{1}{4}u(x(0),0)t$. Therefore

$$u(x,t) = \begin{cases} 4 & x < 4-t \\ ??? & 4-t < x < 4-\frac{3}{4}t \\ 3 & x > 4-\frac{3}{4}t. \end{cases}$$

For the middle interval, we use the equation $x = x(0) - \frac{1}{4}ut$ with $x(0) = 4$ to calculate that $u = \frac{16-4x}{t}$. Therefore

$$u(x,t) = \begin{cases} 4 & x < 4-t \\ \frac{16-4x}{t} & 4-t < x < 4-\frac{3}{4}t \\ 3 & x > 4-\frac{3}{4}t. \end{cases}$$

(e) [3 × 4p] Sketch the graph (u against x) of the solution at times $t = 0$, $t = 1$ and $t = 2$.



Soru 2 (Canonical Forms)**.** Consider the partial differential equation

$$2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = \sin(x^2). \tag{3}$$

(a) [1p] Equation (3) is a

☐ 1st order PDE; ☑ 2nd order PDE; ☐ 3rd order PDE; ☐ 4th order PDE.

(b) [1p] Equation (3) is a

☐ homogeneous PDE; ☑ non-homogeneous PDE.

(c) [1p] Equation (3) is

☑ linear; ☐ non-linear AND quasilinear; ☐ non-linear, but not quasilinear;

(d) [6p] For each $(x, y) \in \mathbb{R}^2$, classify (3) as hyperbolic, parabolic or elliptic.

Since $\Delta = B^2 - 4AC = (-4)^2 - 4 \times 2 \times 2 = 0$, the PDE is parabolic.

(e) [8p] Find the characteristic equation of (3).

The characteristic equation of (3) is $\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{-4 \pm \sqrt{0}}{2 \times 2} = -1$.

(f) [8p] Find the characteristic curve(s) of (3).

Sovling $\frac{dy}{dx} = -1$, we see that $y + x = c$.

(g) [25p] Find a canonical form for (3).
[HINT: I don't want to see x or y in your final answer.]

By part (f), we choose $\xi = y + x$. For a parabolic PDE, we can choose any linearly independent function η. I choose $\eta = x$. We can calculate

$$\xi_x = 1,\ \xi_y = 1,\ \xi_{xx} = 0,\ \xi_{xy} = 0,\ \xi_{yy} = 0,$$

$$\eta_x = 1,\ \eta_y = 0,\ \eta_{xx} = 0,\ \eta_{xy} = 0,\ \eta_{yy} = 0.$$

Therefore

$$\begin{aligned} A^* &= 0 \\ B^* &= 0 \\ C^* &= 2 - 4(0) + 2(0) = 2 \\ D^* &= 0 + 0 + 0 + 0 + 0 = 0 \\ E^* &= 0 + 0 + 0 + 0 + 0 = 0 \\ F^* &= 3 \\ G^* &= \sin(x^2). \end{aligned}$$

Hence

$$2u_{\eta\eta} + 3u = \sin(x^2).$$

The final step is to substitute for x,

$$2u_{\eta\eta} + 3u = \sin(\eta^2).$$

Therefore, a canonical form for (3) is

$$u_{\eta\eta} = -\frac{3}{2}u + \frac{1}{2}\sin(\eta^2).$$

Soru 3 (General Solution)**.** Suppose that $\xi = y - 3x$ and $\eta = y - \frac{x}{3}$.

(a) [20p] **Use the chain rule** (e.g. $u_x = u_\xi \xi_x + u_\eta \eta_x$, etc.) to show that

$$9u_{xx} + 30u_{xy} + 9u_{yy} = -64u_{\xi\eta}.$$

Since

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x = -3u_\xi - \frac{1}{3}u_\eta \\
u_y &= u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta \\
u_{xx} &= (u_x)_\xi \xi_x + (u_x)_\eta \eta_x = (-3u_{\xi\xi} - \frac{1}{3}u_{\eta\xi})(-3) + (-3u_{\xi\eta} - \frac{1}{3}u_{\eta\eta})(-\frac{1}{3}) \\
&= 9u_{\xi\xi} + 2u_{\xi\eta} + \frac{1}{9}u_{\eta\eta} \\
u_{xy} &= (u_x)_\xi \xi_y + (u_x)_\eta \eta_y = (-3u_{\xi\xi} - \frac{1}{3}u_{\eta\xi})(1) + (-3u_{\xi\eta} - \frac{1}{3}u_{\eta\eta})(1) \\
&= -3u_{\xi\xi} - \frac{10}{3}u_{\xi\eta} - \frac{1}{3}u_{\eta\eta} \\
u_{yy} &= (u_y)_\xi \xi_y + (u_y)_\eta \eta_y = (u_{\xi\xi} + u_{\eta\xi})(1) + (u_{\xi\eta} + u_{\eta\eta})(1) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}
\end{aligned}$$

it follows that

$$\begin{aligned}
9u_{xx} + 30u_{xy} + 9u_{yy} &= u_{\xi\xi}\big(81 - 90 + 9\big) + u_{\xi\eta}\big(18 - 100 + 18\big) + u_{\eta\eta}\big(1 - 10 + 9\big) \\
&= -64u_{\xi\eta}.
\end{aligned}$$

Now consider the second order partial differential equation

$$9u_{xx} + 30u_{xy} + 9u_{yy} = -64\sin(2y - 6x). \tag{4}$$

(b) [5p] Equation (4) is a

☑ hyperbolic PDE; ☐ parabolic PDE; ☐ elliptic PDE.

(c) [25p] Find the general solution of (4).
[HINT: I don't want to see ξ or η in your final answer.]

By part (a),

$$-64u_{\xi\eta} = 9u_{xx} + 30u_{xy} + 9u_{yy} = -64\sin(2y - 6x) = -64\sin(2\xi).$$

So

$$u_{\xi\eta} = \sin(2\xi).$$

Integrating (wrt η) gives

$$u_\xi = \int u_{\xi\eta}\, d\eta = \int \sin 2\xi\, d\eta = \eta \sin 2\xi + f(\xi)$$

for some function f. Then integrating (wrt ξ) gives

$$u(\xi, \eta) = \int u_\xi\, d\xi = \int \eta \sin 2\xi + f(\xi)\, d\xi = -\frac{1}{2}\eta \cos 2\xi + F(\xi) + G(\eta)$$

for some functions F and G.

The final step is to change back to x and y: The general solution of (4) is

$$u(x, y) = \frac{1}{2}(\frac{x}{3} - y)\cos(2y - 6x) + F(y - 3x) + G(y - \frac{x}{3}).$$