2016.05.18 **MAT372 K.T.D.D. – Final Sınavın Çözümleri** N. Course

Soru 1 (Fourier Transforms).

(a) [1p] Please write your student number on every page.

(b) [5p] Give the definition of the *convolution* of two functions.

Let f and g be functions. The *convolution* of f and g is

$$f * g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) f(x-\xi)\, d\xi.$$

The Convolution Theorem. *Let f and g be continuous functions. Let $F(\omega)$ and $G(\omega)$ denote the Fourier transforms of f and g respectively. Then*

$$\mathcal{F}\big[f * g\big](\omega) = F(\omega)G(\omega).$$

(c) [19p] Prove the Convolution Theorem.

Since

$$\begin{aligned}
\mathcal{F}^{-1}\big[FG\big](x) &= \int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}\, d\omega \\
&= \int_{-\infty}^{\infty} F(\omega) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)e^{-i\omega\xi}\, d\xi\right] e^{i\omega x}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi) \left[\int_{-\infty}^{\infty} F(\omega)e^{i\omega(x-\xi)}\, d\omega\right] d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)f(x-\xi)\, d\xi \\
&= f * g(x),
\end{aligned}$$

we have that

$$\mathcal{F}\big[f * g\big](\omega) = F(\omega)G(\omega).$$

Soru 2 (Separation of Variables).

[25p] Explain the method of *Separation of Variables* for partial differential equations.

> Imagine that you are explaining the method of *Separation of Variables* to someone who hasn't studied this course. How would you explain it? This question should take you ≈ 25 minutes.
>
> You might like to include:
>
> - the main concepts of this method;
> - an explaination of the *separation constant*
> - an explaination of *eigenvalues* and *eigenfunctions*;
> - an example of your choosing.

[25p] *Değişkenleri Ayırma Yöntemi*ni kısmi türevli diferansiyel denklemleri için açıklayınız.

> Bu dersi almamış birisine *Değişkenleri Ayırma Yöntemi*ni anlatmanız gerektiğini varsayalım. Bu yöntemi nasıl anlatırdınız? Bu soruyu cevaplamak yaklaşık 25 dakikanızı alacaktır.
>
> Bu soruyu cevaplarken aşağıdaki noktalara da yer veriniz:
>
> - bu yöntemin temel kavramları;
> - *ayırma sabitinin* açıklaması;
> - *özdeğer* ve *özişlev*'in açıklamaları;
> - sizin seçeğiniz bir örnek.

The are many possible solutions to this question. Marks will be given generously.

Soru 3 (Characteristics). Consider the PDE

$$\frac{\partial u}{\partial t} + \frac{u}{3}\frac{\partial u}{\partial x} = 0 \tag{1}$$

with the initial condition

$$u(x,0) = \begin{cases} 3 & x < 2 \\ 1 & x > 2. \end{cases} \tag{2}$$

(a) [2p] Replace (1) by a system of 2 ODEs.

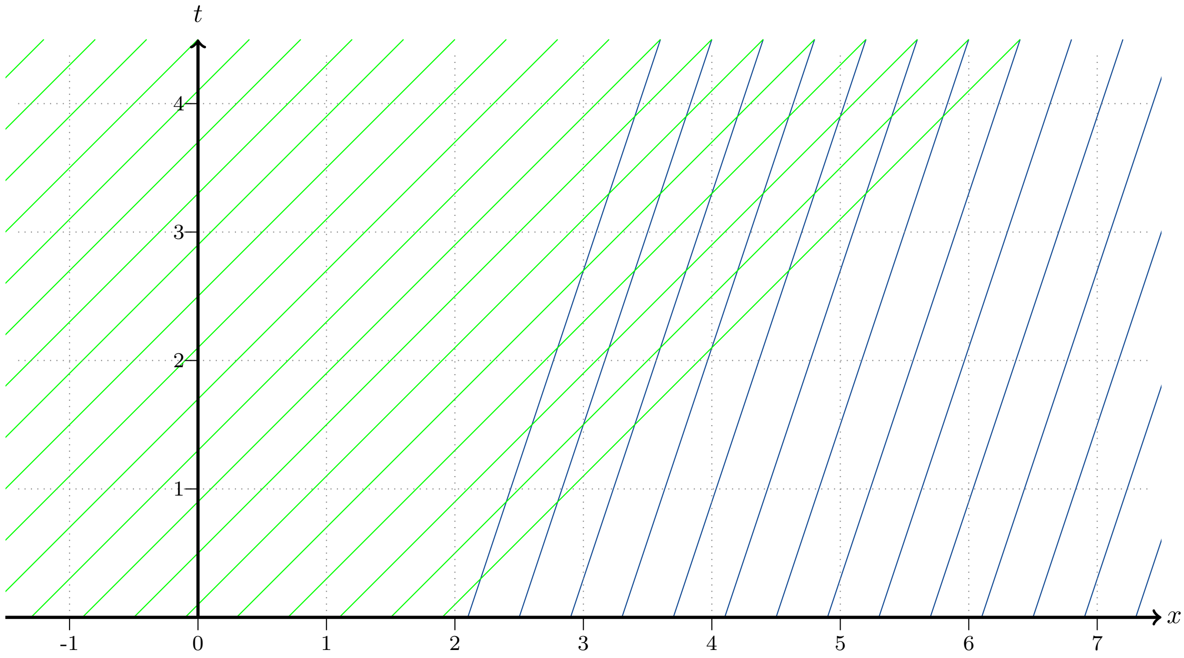
$$\frac{du}{dt} = 0, \qquad \frac{dx}{dt} = \frac{u}{3}$$

(b) [7p] Plot the characteristics (t against x) for this problem.

The solution of $u' = 0$ is $u(x,t) = u(x(0),0)$, and the solution of $x' = \frac{1}{3}u(x(0),0)$ is

$$x(t) = x(0) + \frac{1}{3}u(x(0),0)t = \begin{cases} x(0) + t & x(0) < 2 \\ x(0) + \frac{t}{3} & x(0) > 2. \end{cases}$$

Thus



(c) [1p] Does this problem have *fan-like characteristics*, *shock wave characteristics*, *neither* or *both*? [Mark ☑ only one box.]

☐ fan-like characteristics, ☑ shock wave characteristics, ☐ neither, ☐ both.

(d) [9p] Solve

$$\frac{\partial u}{\partial t} + \frac{u}{3}\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 3 & x < 2 \\ 1 & x > 2. \end{cases}$$

We can see from part (b) that there is a shock wave starting at $x_0 = 2$.

Since

$$[u] = \lim_{x \searrow 2} u(x,0) - \lim_{x \nearrow 2} u(x,0) = 1 - 3 = -2,$$

$$q(u) = \tfrac{1}{6}u^2$$

(because $\frac{dq}{du} = \frac{u}{3}$) and

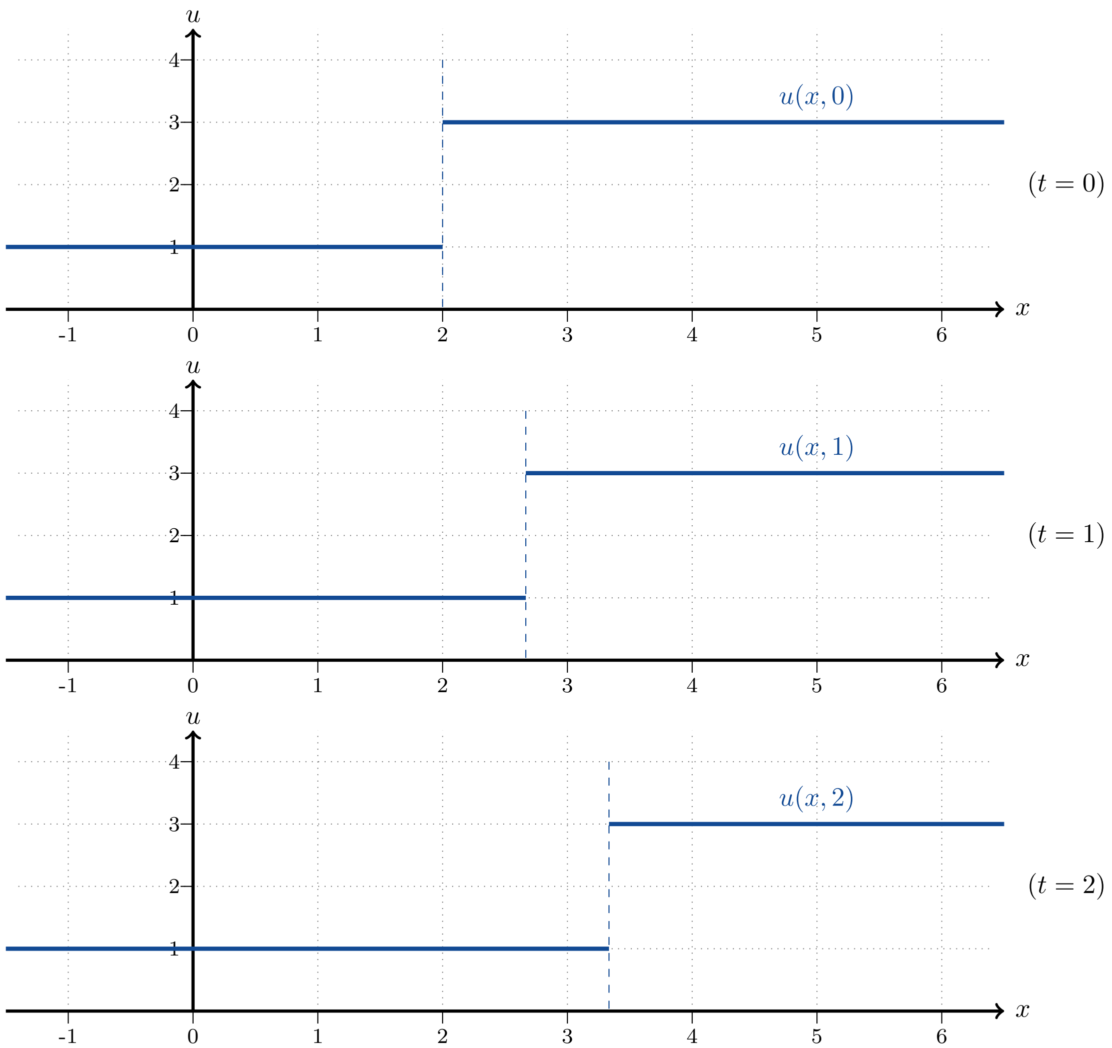
$$[q] = \lim_{x \searrow 2} q(u(x,0)) - \lim_{x \nearrow 2} q(u(x,0)) = \tfrac{1}{6}\cdot 1^2 - \tfrac{1}{6}\cdot 3^2 = -\tfrac{8}{6} = -\tfrac{4}{3},$$

the shock characteristic is obtained by solving $\frac{dx_s}{dt} = \frac{[q]}{[u]} = \frac{2}{3}$. So $x_s = \frac{2}{3}t + x_s(0) = \frac{2}{3}t + 2$. This is where the behaviour of the solution changes.

Therefore the solution is

$$u(x,t) = \begin{cases} 1 & x < x_s(t) \\ 3 & x > x_s(t) \end{cases} = \begin{cases} 1 & x < \frac{2}{3}t + 2 \\ 3 & x > \frac{2}{3}t + 2 \end{cases} = \begin{cases} 1 & x - \frac{2}{3}t < 2 \\ 3 & x - \frac{2}{3}t > 2. \end{cases}$$

(e) [3×2p] Sketch the graph (u against x) of the solution at times $t = 0$, $t = 1$ and $t = 2$.



Soru 4 (Canonical Forms). Consider the second order partial differential equation

$$u_{xx} + y^2 u_{yy} = y^2 \tag{3}$$

for $y \neq 0$.

(a) [1p] Calculate the discriminant $\Delta(x, y)$ of (3).

Clearly
$$\Delta = B^2 - 4AC = 0 - 4 \times 1 \times y^2 = -4y^2 < 0$$

(b) [2p] If $y \neq 0$, equation (3) is a/an

☐ hyperbolic PDE; ☐ parabolic PDE; ☑ elliptic PDE.

(c) [2p] Find the characteristic equation of (3).

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{0 \pm \sqrt{-4y^2}}{2} = \pm iy$$

(d) [5p] Find the characteristic curve(s) of (3).

$$\log y = \pm ix + c$$

$$u_{xx} + y^2 u_{yy} = y^2 \qquad (y \neq 0) \tag{3}$$

(e) [15p] Find a canonical form for (3).
[HINT: x and y MUST NOT appear in your final answer. I want to only see u, ξ, η; or only see u, α, β.]

Let $\xi = \log y + ix$ and $\eta = \log y - ix$. Then let $\alpha = \operatorname{Re}\xi = \log y$ and $\beta = \operatorname{Im}\xi = x$. $\boxed{3}$
We calculate that

$$\begin{aligned} \alpha_x &= 0 & \beta_x &= 1 \\ \alpha_y &= \tfrac{1}{y} & \beta_y &= 0 \\ \alpha_{xx} &= 0 & \beta_{xx} &= 0 \qquad \boxed{3} \\ \alpha_{xy} &= 0 & \beta_{xy} &= 0 \\ \alpha_{yy} &= -\tfrac{1}{y^2} & \beta_{yy} &= 0 \end{aligned}$$

and that

$$\begin{aligned} A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 0 + 0 + y^2\left(\tfrac{1}{y}\right)^2 = 1 \\ B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y + \beta_x) + 2C\alpha_y\beta_y = 0 + 0 + 0 = 0 \\ C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 = 1 + 0 + 0 = 1 \\ D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y = 0 + 0 + y^2\left(-\tfrac{1}{y^2}\right)^2 + 0 + 0 = -1 \qquad \boxed{4} \\ E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y = 0 + 0 + 0 + 0 + 0 = 0 \\ F^{**} &= F = 0 \\ G^{**} &= y^2 = e^{2\alpha}. \qquad \boxed{3} \end{aligned}$$

Therefore a canonical form for (3) is

$$u_{\alpha\alpha} + u_{\beta\beta} - u_\alpha = e^{2\alpha}. \qquad \boxed{2}$$

Soru 5 (Fourier Series).

(a) [5p] Let $n, m \in \mathbb{N}$ such that $n \neq m$. Show that the functions $\sin\frac{n\pi x}{2}$ and $\cos\frac{m\pi x}{2}$ are orthogonal on $[-2, 2]$.

Note that

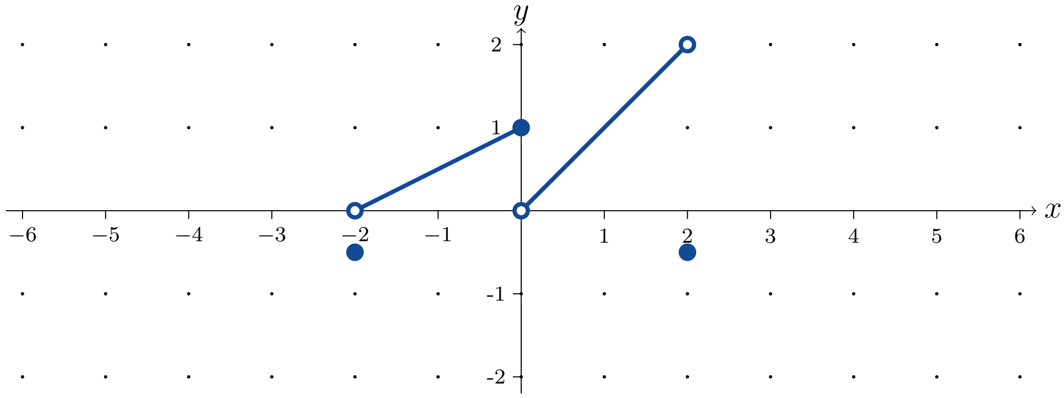
$$\begin{aligned} \left\langle \sin\frac{n\pi x}{2}, \cos\frac{m\pi x}{2} \right\rangle &= \int_{-2}^{2} \sin\frac{n\pi x}{2}\cos\frac{m\pi x}{2}\,dx \\ &= \frac{1}{2}\int_{-2}^{2} \sin\frac{(n+m)\pi x}{2} + \sin\frac{(n-m)\pi x}{2}\,dx \\ &= 0 \end{aligned}$$

because sine is an odd function. Therefore $\sin\frac{n\pi x}{2}$ and $\cos\frac{m\pi x}{2}$ are orthogonal on $[-2, 2]$.

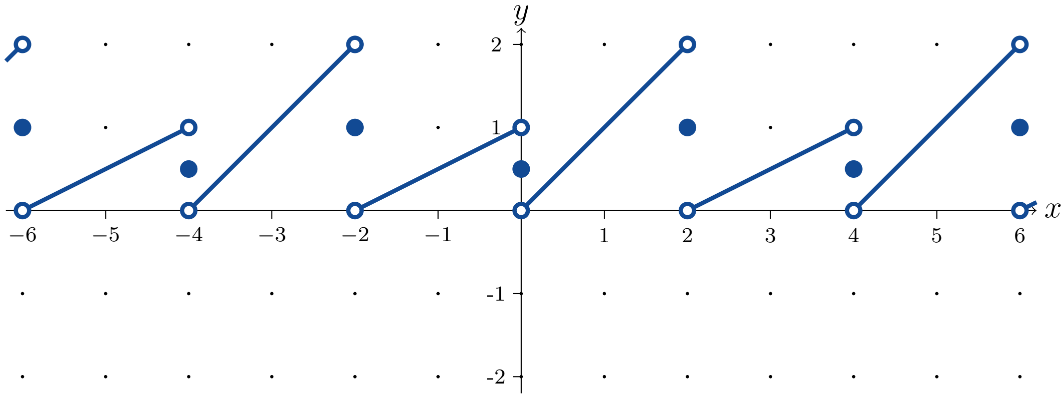
Define the function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\frac{1}{2} & x = -2,\ x = 2, \\ \frac{x}{2} + 1 & -2 < x \leq 0 \\ x & 0 < x < 2. \end{cases} \tag{4}$$

(b) [1p] Sketch f.



(c) [6p] Sketch the Fourier Series of f.



$$f(x) = \begin{cases} -\frac{1}{2} & x = -2,\ x = 2, \\ \frac{x}{2} + 1 & -2 < x \leq 0 \\ x & 0 < x < 2. \end{cases} \tag{5}$$

(d) [13p] Calculate the coefficients $a_0, a_1, a_2, a_3, a_4, \ldots$ of the Fourier Series of f on $[-2, 2]$. [You do not need to calculate $b_k = -\frac{1}{k\pi}\left(1 + (-1)^k \times 2\right)$.]

We calculate that

$$a_0 = \frac{1}{2}\int_{-2}^{0} \frac{x}{2} + 1\ dx + \frac{1}{2}\int_{0}^{2} x\ dx = \frac{1}{2}\left[\frac{1}{4}x^2 + x\right]_{-2}^{0} + \frac{1}{2}\left[\frac{1}{2}x^2\right]_{0}^{2} = -\frac{1}{2} + 1 + 1 = \frac{3}{2}$$

$$\begin{aligned} a_k &= \frac{1}{2}\int_{-2}^{0} \left(\frac{x}{2} + 1\right)\cos\frac{k\pi x}{2}\ dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{k\pi x}{2}\ dx \\ &= \frac{1}{2}\left[\frac{x}{k\pi}\sin\frac{k\pi x}{2} + \frac{2}{(k\pi)^2}\cos\frac{k\pi x}{2} + \frac{2}{k\pi}\sin\frac{k\pi x}{2}\right]_{-2}^{0} \\ &\quad + \frac{1}{2}\left[\frac{2x}{k\pi}\sin\frac{k\pi x}{2} + \left(\frac{2}{k\pi}\right)^2\cos\frac{k\pi x}{2}\right]_{0}^{2} \\ &= \frac{1}{2}\left(0 + \frac{2}{(k\pi)^2} + 0 - 0 - \frac{2}{(k\pi)^2}(-1)^k - 0\right) + \frac{1}{2}\left(0 + \left(\frac{2}{k\pi}\right)^2(-1)^k - 0 - \left(\frac{2}{k\pi}\right)^2\right) \\ &= \frac{1}{(k\pi)^2} - \frac{1}{(k\pi)^2}(-1)^k + \frac{2}{(k\pi)^2}(-1)^k - \frac{2}{(k\pi)^2} \\ &= \frac{1}{(k\pi)^2}\left((-1)^k - 1\right). \end{aligned}$$