OKAN ÜNİVERSİTESİ
MÜHENDİSLİK-MİMARLIK FAKÜLTESİ
MÜHENDİSLİK TEMEL BİLİMLERİ BÖLÜMÜ

2016.03.30 **MAT372 K.T.D.D. – Ara Sınavın Çözümleri** N. Course

Soru 1 (Characteristics). Consider the PDE

$$\frac{\partial u}{\partial t} + \frac{u}{3}\frac{\partial u}{\partial x} = 0 \tag{1}$$

with the initial condition

$$u(x,0) = \begin{cases} 1 & x < 2 \\ 3 & x > 2. \end{cases} \tag{2}$$

(a) [1p] Please write your student number at the top right of this page.

(b) [5p] Replace (1) by a system of 2 ODEs.

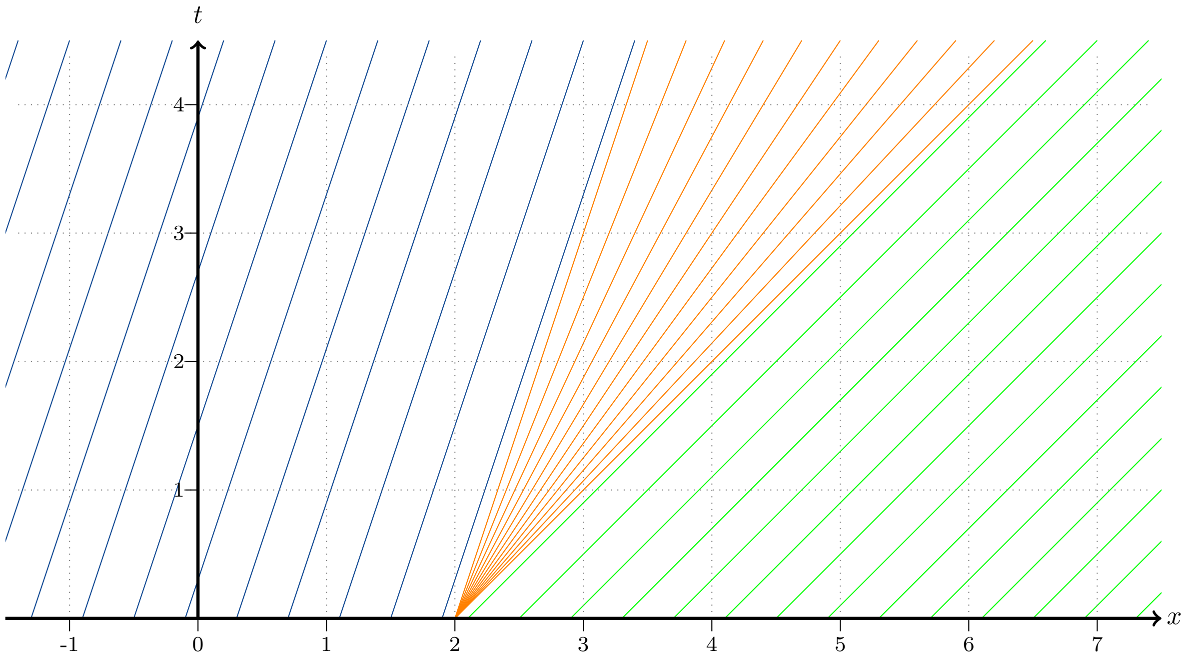
$\frac{du}{dt} = 0$, $\frac{dx}{dt} = \frac{u}{3}$

(c) [15p] Plot the characteristics (t against x) for this problem.

The solution of $u' = 0$ is $u(x,t) = u(x(0),0)$, and the solution of $x' = \frac{1}{3}u(x(0),0)$ is

$$x(t) = x(0) + \frac{1}{3}u(x(0),0)t = \begin{cases} x(0) + \frac{t}{3} & x(0) < 2 \\ x(0) + t & x(0) > 2. \end{cases}$$

Thus



(d) [2p] Does this problem have *fan-like characteristics*, *shock wave characteristics*, *neither* or *both*? [Mark ☑ only one box.]

☑ fan-like characteristics ☐ shock wave characteristics ☐ neither ☐ both

(e) [15p] Solve

$$\frac{\partial u}{\partial t} + \frac{u}{3}\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 1 & x < 2 \\ 3 & x > 2. \end{cases}$$

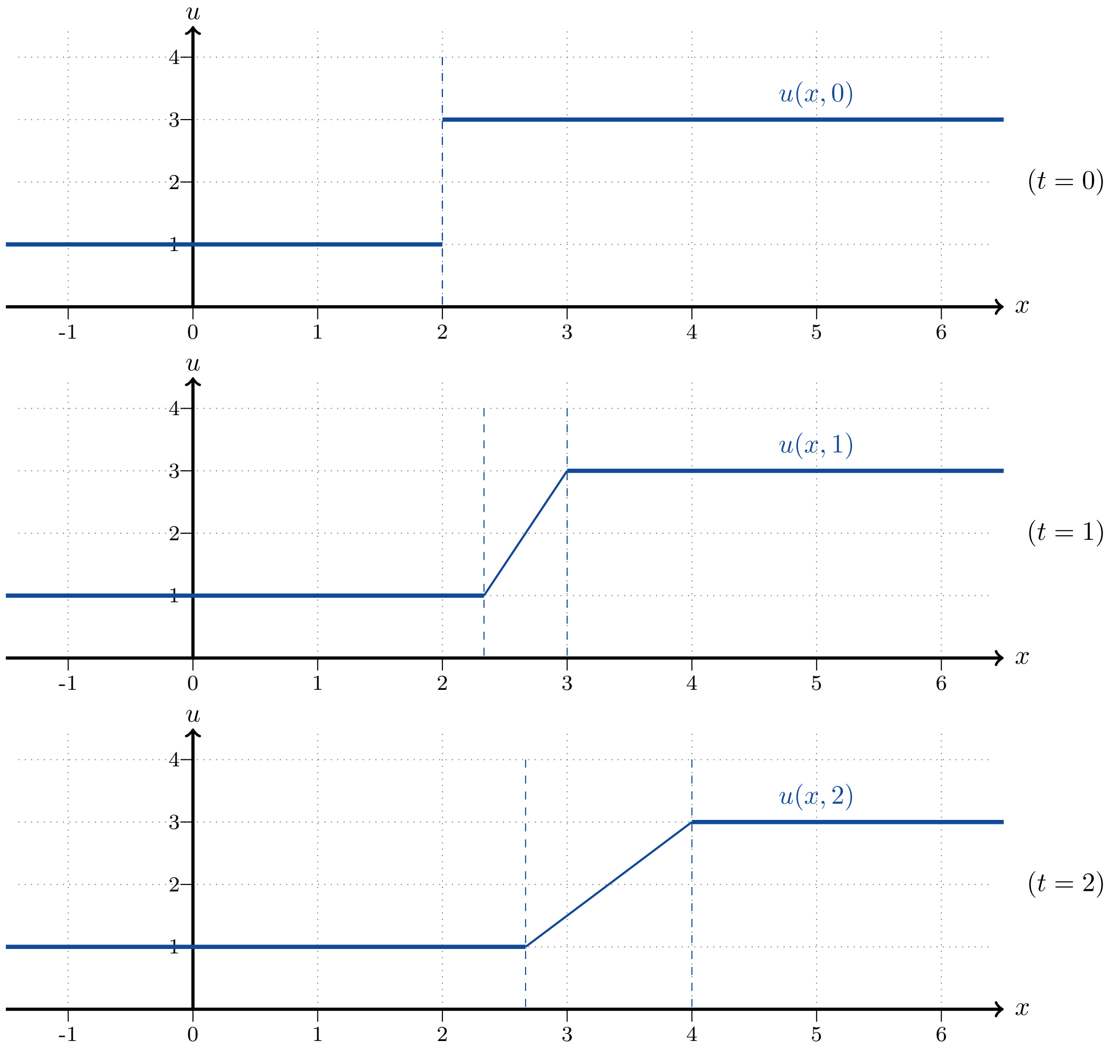
As above, $u(x,t) = u(x(0),0)$ and $x(t) = x(0) + \frac{1}{3}u(x(0),0)t$. Therefore

$$u(x,t) = \begin{cases} 1 & x < 2 + \frac{t}{3} \\ ??? & 2 + \frac{t}{3} < x < 2 + t \\ 3 & x > 2 + t. \end{cases}$$

For the middle interval, we use the equation $x = x(0) + \frac{1}{3}ut$ with $x(0) = 2$ to calculate that $u = \frac{3x-6}{t}$. Therefore

$$u(x,t) = \begin{cases} 1 & x < 2 + \frac{t}{3} \\ \frac{3x-6}{t} & 2 + \frac{t}{3} < x < 2 + t \\ 3 & x > 2 + t. \end{cases}$$

(f) [3 × 4p] Sketch the graph (u against x) of the solution at times $t = 0$, $t = 1$ and $t = 2$.



Soru 2 (Change of variables). Consider the second order, linear PDE

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y). \tag{3}$$

Suppose that $\xi(x,y)$ and $\eta(x,y)$ are twice continuously differentiable functions of x and y.

Using the chain rule, we can calculate that

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi \xi_x + u_\eta \eta_x.$$

(a) [1p] Please write your student number at the top right of this page.

(b) [24p] Use the chain rule to find formulae for u_y, u_{xy}, u_{xx} and u_{yy} in terms of $u_{\xi\xi}$, $u_{\xi\eta}$, $u_{\eta\eta}$, u_ξ, u_η, u, ξ_{xx}, ξ_{xy}, ξ_{yy} ξ_x, ξ_y, η_{xx}, η_{xy}, η_{yy}, η_x and η_y.

As above, $u_y = u_\xi \xi_y + u_\eta \eta_y$. $\boxed{3}$ Therefore

$$\begin{aligned} u_{xy} &= (u_x)_y = (u_\xi \xi_x + u_\eta \eta_x)_y \\ &= (u_\xi)_y \xi_x + u_\xi \xi_{xy} + (u_\eta)_y \eta_x + u_\eta \eta_{xy} \\ &= (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + u_\xi \xi_{xy} + (u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y)\eta_x + u_\eta \eta_{xy} \\ &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}. \ \boxed{7} \end{aligned}$$

By replacing all the y's with x's, it is easy to see that

$$u_{xx} = u_{\xi\xi}\xi_x\xi_x + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx}. \ \boxed{7}$$

Similarly, by replacing all the x's with y's, we have

$$u_{yy} = u_{\xi\xi}\xi_y\xi_y + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y\eta_y + u_\xi \xi_{yy} + u_\eta \eta_{yy}. \ \boxed{7}$$

By substituting your answers to part (b) into (3), we obtain an equation of the form

$$A^*(\xi,\eta)u_{\xi\xi} + B^*(\xi,\eta)u_{\xi\eta} + C^*(\xi,\eta)u_{\eta\eta} + D^*(\xi,\eta)u_\xi + E^*(\xi,\eta)u_\eta + F^*(\xi,\eta)u = G^*(\xi,\eta). \quad (4)$$

(c) [25p] Show that

$$B^*(\xi,\eta) = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y.$$

We are only interested in the $u_{\xi\eta}$ terms. From Au_{xx}, we obtain $2Au_{\xi\eta}\xi_x\eta_x$. From Bu_{xy}, we obtain $Bu_{\xi\eta}(\xi_x\eta_y+\xi_y\eta_x)$. From Cu_{yy}, we obtain $2Cu_{\xi\eta}\xi_y\eta_y$. Note that Du_x+Eu_y+Fu-G will not give us any $u_{\xi\eta}$ terms.
Therefore

$$\begin{aligned} &Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu - G \\ &\quad = (\text{some } u_{\xi\xi} \text{ terms}) + \big(2Au_{\xi\eta}\xi_x\eta_x + Bu_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + 2Cu_{\xi\eta}\xi_y\eta_y\big) \\ &\quad\quad + (\text{some } u_{\eta\eta} \text{ terms}) + (\text{some smaller derivative terms}). \end{aligned}$$

It follows that we must have $B^*(\xi,\eta) = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$.

Soru 3 (General Solution). Consider the partial differential equation

$$u_{xx} - \frac{1}{c^2}u_{yy} = -\frac{4}{c^2}y - \frac{4}{c^3}x \quad (5)$$

where $c \neq 0$ is a constant.

(a) [1p] Please write your student number at the top right of this page.

(b) [1p] Equation (5) is a

☐ 1^{st} order PDE; ☑ 2^{nd} order PDE; ☐ 3^{rd} order PDE; ☐ 4^{th} order PDE.

(c) [1p] Equation (5) is a

☐ homogeneous PDE; ☑ non-homogeneous PDE.

(d) [1p] Equation (5) is

☑ linear; ☐ non-linear AND quasilinear; ☐ non-linear, but not quasilinear.

The discrimant of (5) is $\Delta = B^2 - 4AC = \frac{4}{c^2} > 0$.

(e) [1p] Equation (5) is a/an

☑ hyperbolic PDE; ☐ parabolic PDE; ☐ elliptic PDE.

The characteristic equation of (5) is

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \pm\frac{1}{c}$$

which has solutions $y + \frac{x}{c} = (\text{constant})$ and $y - \frac{x}{c} = (\text{constant})$.

(f) [22p] Find a canonical form for (5).

Let $\xi = y + \frac{x}{c}$ and $\eta = y - \frac{x}{c}$. Then

$$\begin{aligned} \xi_x &= \frac{1}{c} & \eta_x &= -\frac{1}{c} \\ \xi_y &= 1 & \eta_y &= 1 \\ \xi_{xx} &= 0 & \eta_{xx} &= 0 \\ \xi_{xy} &= 0 & \eta_{xy} &= 0 \\ \xi_{yy} &= 0 & \eta_{yy} &= 0 \end{aligned}$$

and

$$\begin{aligned} A^* &= 1\left(\frac{1}{c}\right)^2 + 0 - \frac{1}{c^2}(1)^2 = 0 \\ B^* &= 2(1)\left(\frac{1}{c}\right)\left(-\frac{1}{c}\right) + 0 + 2(-\frac{1}{c^2})(1)(1) = -\frac{4}{c^2} \\ C^* &= 1\left(-\frac{1}{c}\right)^2 + 0 - \frac{1}{c^2}(1)^2 = 0 \\ D^* &= 0 + 0 + 0 + 0 + 0 = 0 \\ E^* &= 0 + 0 + 0 + 0 + 0 = 0 \\ F^* &= 0 \\ G^* &= G = -\frac{4}{c^2}y - \frac{4}{c^3}x \end{aligned}$$

Since $-\frac{4}{c^2}y - \frac{4}{c^3}x = -\frac{4}{c^2}\xi$, we have that $-\frac{4}{c^2}u_{\xi\eta} = -\frac{4}{c^2}\xi$. It follows that a canonical form for (5) is

$$u_{\xi\eta} = \xi.$$

(g) [23p] Find the general solution to

$$u_{xx} - \frac{1}{c^2}u_{yy} = -\frac{4}{c^2}y - \frac{4}{c^3}x$$

where $c \neq 0$ is a constant.

First we need to solve
$$u_{\xi\eta} = \xi.$$
Integrating with respect to η gives
$$u_\xi = \int u_{\xi\eta}\ d\eta = \int \xi\ d\eta = \xi\eta + f(\xi)$$
for some function f. Then integrating with respect to ξ gives
$$u(\xi,\eta) = \int u_\xi\ d\xi = \int \xi\eta + f(\xi)\ d\xi = \frac{1}{2}\xi^2\eta + F(\xi) + G(\eta)$$
for some functions F and G.

Changing back to x and y, we obtain
$$u(x,y) = \frac{1}{2}\left(y + \frac{x}{c}\right)^2\left(y - \frac{x}{c}\right) + F\left(y + \frac{x}{c}\right) + G\left(y - \frac{x}{c}\right).$$