f-Harmonic Maps

by

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Thesis submitted for the degree of Doctor of Philosophy.

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Declaration

The contents of this thesis are, to the best of my knowledge, the original work of the author except where stated otherwise. No part of this work has been previously offered for publication, nor has it been submitted for a degree at any other university or institution.

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Abstract[1]

A study of f-harmonic maps and f-harmonic heat flow.

The first chapter gives the definitions of f-harmonic maps and heat flow; namely the critical points of the energy functional $E_f(u) := \frac{1}{2}\int_{\mathcal{M}} f|\nabla u|^2 \, d\mathcal{M}$ (for a compact surface $\mathcal{M}$) and the L^2–gradient flow of this energy. The first and second variations of E_f are calculated, and previously known f-harmonic results are stated.

In section 2.1, we see that a smooth f-harmonic map of finite energy, defined on the disc minus one point, smoothly extends to the whole disc. Section 2.2 investigates: Given an f-harmonic map u and another function f_1 "close" to f, is there an f_1-harmonic map "close" to u? By the Implicit Function Theorem, the answer is "yes if" we have a hypothesis on the Jacobi Operator.

Chapter 3 studies the f-harmonic map heat flow, and extends the existence and "bubbling" result of Struwe to f-harmonic map heat flow.

A result of Lemaire (*"every harmonic map from a compact, contractible surface, with constant boundary data must be constant"*) is considered in chapter 4. The f-harmonic map heat flow yields examples demonstrating that this result doesn't extend completely to f-harmonic maps. There is however an analogue for certain f. A particular sequence of f_n-harmonic maps ($D \to S^2$, $\partial D \mapsto \{0\}$) is studied, with $f_n \to 1$, to try to see how an f-harmonic map might need to be varied (so as to remain f-harmonic) if the f is "flattened" towards $f \equiv 1$.

The main result of this thesis is given in chapter 5: Every infinite time 'bubble' point in $\operatorname{int}\mathcal{M}$, must be a critical point of f. A hypothesized refinement, that infinite time bubbles only form at minima of f, is false (chapter 6). However bubbles forming at finite times can form at non-critical points of f.

Finally, chapter 7 briefly describes a possible application of this theory.

[1]There is an extended summary starting on page 83

Chapter 1

Introduction

1.1 Basic definitions

Let $(\mathcal{M}, g)$ be a compact Riemannian manifold (with or without boundary). Let $(\mathcal{N}, h)$ be a compact Riemannian manifold without boundary, embedded isometrically in $\mathbb{R}^N$. This embedding is always possible by the Nash Embedding Theorem. Let $f : \mathcal{M} \to (0, \infty)$ be a smooth function. By compactness, f is bounded above, and below by a strictly positive number, say $0 < A \leq f(x) \leq B$.

Definition 1.1.1 (f-harmonic energy). Let $u \in W^{1,2}(\mathcal{M}; \mathcal{N})$. The *$f$-harmonic energy functional* is defined to be

$$E_f(u) = \frac{1}{2} \int_{\mathcal{M}} f(x) |\nabla u|^2 d\mathcal{M} \tag{1.1.1}$$

where $d\mathcal{M} = \sqrt{g} dx^1 \wedge \ldots \wedge dx^m$ ($\dim \mathcal{M} = m$) and $\sqrt{g}$ denotes $\big(\det g_{\alpha\beta}\big)^{\frac{1}{2}}$. For consistency of notation, we denote the *harmonic energy* by E_1.

Definition 1.1.2 (tubular neighbourhood/nearest point projection). For $\rho > 0$, define a tubular neighbourhood of $\mathcal{N}$ by

$$V_\rho \mathcal{N} := \{z \in \mathbb{R}^N : d(z, \mathcal{N}) < \rho\} \subset \mathbb{R}^N. \tag{1.1.2}$$

Here $d(z, \mathcal{N})$ denotes of course $\inf\{|z - x|_{\mathbb{R}^N} : x \in \mathcal{N}\}$. Choosing $\rho > 0$ sufficiently small, we may let $P : V_\rho \mathcal{N} \to \mathcal{N}$ denote "nearest point" projection. P is well-defined and smooth – see e.g. [Sim91, §2.12.3].

Definition 1.1.3 (admissible variation). An *admissible variation* of u, is a family of maps $u_s := P \circ (u + s\phi)$, for some $\phi \in C_c^\infty(\mathcal{M}, \mathbb{R}^N)$ and for small $|s|$. Notice that $u_0 = u$ and that $u_s \equiv u$ in a neighbourhood of $\partial \mathcal{M}$.

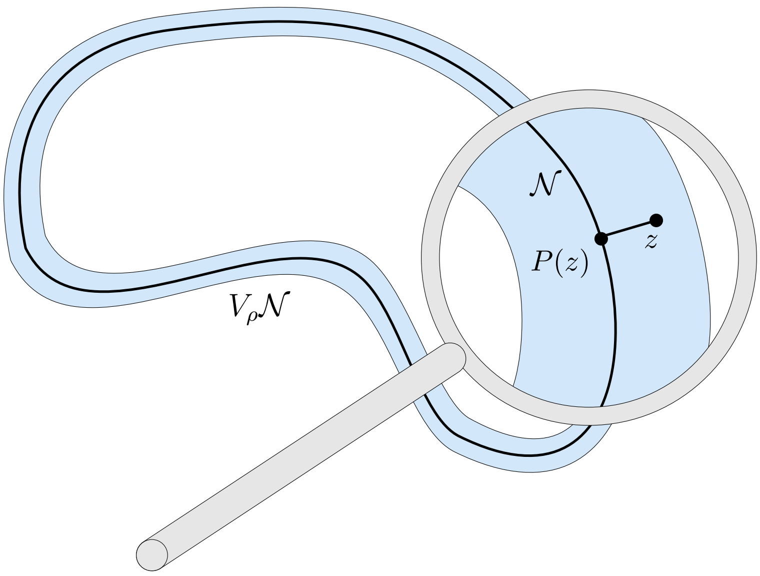

Figure 1.1: The tubular neighbourhood $V_\rho\mathcal{N}$.

Remark. If $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ and $\phi \in C_c^\infty(\mathcal{M};\mathbb{R}^N)$, then $P \circ (u + s\phi) \in W^{1,2}(\mathcal{M};\mathcal{N})$ for sufficiently small $|s|$ [Sim91, §2.2], and $\nabla P \circ (u + s\phi)$ is differentiable with respect to s. Hence $E_f(P \circ (u + s\phi))$ is also differentiable. We can then define:

Definition 1.1.4 (f-harmonic). A map $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ is said to be (weakly) *f-harmonic* if, for any (admissible) variation u_s, of u, we have that

$$\frac{d}{ds} E_f(u_s)\bigg|_{s=0} = 0. \tag{1.1.3}$$

Remark. A *harmonic* map satisfies this definition as a *1-harmonic* map.

Before we proceed, it is worth clearing up any possible confusion with the name *f-harmonic*. Our "f-harmonic maps" should not be confused with, so called, *F-harmonic maps* (see e.g. [Ara01] or [Leu96]) which are critical points of the energy functional

$$E_F(u) = \int_{\mathcal{M}} F\left(\frac{1}{2}|\nabla u|^2\right) d\mathcal{M}$$

for a non-negative, strictly increasing, C^2 function F on the interval $[0, \infty)$.

Neither should an f-harmonic map be confused with a p-harmonic map, namely a critical point of the energy functional

$$E_p(u) = \frac{1}{2} \int_{\mathcal{M}} |\nabla u|^p d\mathcal{M}.$$

Specifically, in the language of p-harmonic maps, an "ordinary" *harmonic* map could be referred to as a 2-harmonic map and the associated energy as E_2. In this work, we will refer to harmonic maps as 1-harmonic and denote the harmonic energy by E_1. Of course a 1-harmonic map (our terminology) is also a λ-harmonic map for any constant $\lambda > 0$.

1.2 A comment on the case of dimension greater than 2.

Definition 1.2.1 (Warped Product). Given Riemannian manifolds (M_1, g_1) and (M_2, g_2), and a smooth function $f : \mathcal{M}_1 \to (0, \infty)$, the *warped product* $M_1 \times_f M_2$ is defined to be the manifold $M := M_1 \times M_2$ together with the metric $g = g_1 + f g_2$.

When $\dim \mathcal{M} \neq 2$, any f-harmonic map $(\mathcal{M}, g) \to (\mathcal{N}, h)$ is a harmonic map $(\mathcal{M}, f^{\frac{2}{m-2}} g) \to (\mathcal{N}, h)$ ([EL78, §10.20] or [EL95, §10.20]). However when $\dim \mathcal{M} = 2$, a conformal change of metric keeps an f-harmonic map, f-harmonic (for the same f).

But if $\dim \mathcal{M} = 2$, we may consider an f-harmonic map $\mathcal{M} \to \mathcal{N}$ as a *harmonic* map on a certain higher dimensional manifold (perhaps $\mathcal{M} \times_{f^2} S^1$). This motivates one to study only f-harmonic maps from a surface. An interesting example of an f-harmonic map, or perhaps of an f-harmonic heat flow (see below), on a surface *may* (for example, with boundary singularity) immediately give an interesting harmonic example on some higher dimensional manifold.

Moreover, f-harmonic heat flow from a surface *may* be a useful tool to find a harmonic heat flow (from e.g. a three dimensional domain) where the set of points $\bar{z}$ with the property "*on every neighbourhood of $\bar{z}$, the flow u does not have uniformly (with t) bounded derivative*" is two dimensional. See chapter 7 for an idea for such an approach.

From here onwards, let $(\mathcal{M}, g)$ be a compact surface.

1.3 Character of an f-harmonic map

Let $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$ be f-harmonic. The first order Taylor approximation for $P : \mathbb{R}^N \to \mathbb{R}^N$ is

$$P(u + s\phi) = u + sdP_u(\phi) + R$$

where

$$R = s^2 \sum_{i,j=1}^{N} \int_0^1 \frac{(t-1)^2}{2} \left(\frac{\partial^2 P}{\partial y^i \partial y^j}\big(u + st\phi\big) \right) \phi^i \phi^j \, dt$$

for coordinates $(y_1, \ldots, y^N) \in \mathbb{R}^N$. This term makes sense as P and ϕ are smooth and u is $W^{1,2}$. Write $D_\alpha = \frac{\partial}{\partial x_\alpha}$. Then (c.f. [Sim91, §2.2])

$$D_\alpha u_s = D_\alpha u + s\Big[dP_u(D_\alpha \phi) + \operatorname{Hess} P_u(\phi, D_\alpha u)\Big] + D_\alpha R. \tag{1.3.1}$$

Notice that $D_\alpha R$ is differentiable with respect to s, and that

$$\frac{\partial}{\partial s} D_\alpha R\bigg|_{s=0} = 0. \tag{1.3.2}$$

We calculate (as in [Sim91, §2.2]) that

$$\begin{aligned} 0 &= \frac{d}{ds} E_f(u_s)\bigg|_{s=0} \\ &= \frac{d}{ds}\frac{1}{2} \int_{\mathcal{M}} f(x) |\nabla u_s|^2 d\mathcal{M}\bigg|_{s=0} \\ &= \int_{\mathcal{M}} f \sum_{\alpha=1}^{2} \big\langle D_\alpha u, dP_u(D_\alpha \phi) + \operatorname{Hess} P_u(\phi, D_\alpha u)\big\rangle d\mathcal{M} \qquad (1.3.3) \\ &= \int_{\mathcal{M}} f \sum_{\alpha=1}^{2} \langle D_\alpha u, D_\alpha \phi\rangle + \langle \phi, \operatorname{Hess} P_u(D_\alpha u, D_\alpha u)\rangle \, d\mathcal{M} \\ &= \int_{\mathcal{M}} \langle \nabla \phi, f \nabla u\rangle - \langle \phi, f A(u)(\nabla u, \nabla u)\rangle \, d\mathcal{M} \end{aligned}$$

by Theorem 1 of §2.12.3 in [Sim91]. Suppose now that u is a "classical" f-harmonic map [Lic70] – that is a C^2 map satisfying Definition 1.1.4. Then we may integrate by parts to obtain

$$0 = \frac{d}{ds} E_f(u_s)\bigg|_{s=0} = -\int_{\mathcal{M}} \langle \phi, f\Delta_{\mathcal{M}} u + fA(u)(\nabla u, \nabla u) + \nabla f * \nabla u \rangle \, d\mathcal{M}$$
$$= -\int_{\mathcal{M}} \langle \phi, f(x)\tau_1(u) + \nabla f * \nabla u \rangle \, d\mathcal{M}$$

where $\nabla f * \nabla u := \langle \nabla f, \nabla u^i \rangle \frac{\partial}{\partial u^i} \in T_u\mathcal{N}$. Here $\Delta_{\mathcal{M}}$ denotes the Laplace-Beltrami operator

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\beta} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \right),$$

τ_1 denotes the usual 1-harmonic tension and A denotes the second fundamental form of the embedding of $\mathcal{N}$ in $\mathbb{R}^N$. We will often drop the " $_{\mathcal{M}}$" subscript and write just Δ. This calculation is repeated in implicit coordinates, in §1.6.2, but for now we have the following;

Lemma 1.3.1 (Euler-Lagrange equation for E_f). *Let $u \in C^2(\mathcal{M};\mathcal{N})$. The following are equivalent:*

(i) u is f-harmonic;

(ii) $f\Delta_{\mathcal{M}} u + fA(u)(\nabla u, \nabla u) + \nabla f * \nabla u = 0$;

(iii) the harmonic *tension of u is $\tau_1(u) = -\frac{1}{f}\nabla f * \nabla u$;*

(iv) $\operatorname{div}(f\nabla u)$ *is perpendicular to $T\mathcal{N}$.*

Similarly if $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ then: u is weakly f-harmonic if and only if u satisfies part (ii) above, weakly (with test functions $\phi \in C_c^\infty(\mathcal{M};\mathbb{R}^N)$). If $\partial\mathcal{M}$ is "nice", we can slightly generalize this statement, to one which we will need later:

Lemma 1.3.2. *Suppose that $\partial\mathcal{M}$ is C^1. The map $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ is weakly f-harmonic if and only if*

$$0 = \int_{\mathcal{M}} \langle \nabla\phi, f\nabla u \rangle - \langle \phi, fA(u)(\nabla u, \nabla u) \rangle \, d\mathcal{M}, \tag{1.3.4}$$

for all $\phi \in W_0^{1,2} \cap C^0(\mathcal{M};\mathbb{R}^N)$.

Proof. Let $\phi \in W_0^{1,2} \cap C^0(\mathcal{M};\mathbb{R}^N)$. There exists a sequence $\phi_n \in C_c^\infty(\mathcal{M};\mathbb{R}^N)$ such that $\phi_n \to \phi$ in $W^{1,2} \cap L^\infty$ by the following argument: Take a cut-off function $\eta \in C^\infty([0,\infty),[0,1])$ such that $\eta(t)=0$ for $t\in[0,1]$ and $\eta(t)=1$ for $t\geq 2$. Now define $\varsigma_\delta \in C_c^\infty(\mathcal{M},[0,1])$ by

$$\varsigma_\delta(x) = \eta\left(\frac{1}{\delta}\, d(x,\partial\mathcal{M})\right).$$

Then $\varsigma_\delta\phi \to \phi$ in $W^{1,2}\cap L^\infty$ as $\delta \to 0$.

For each $\delta > 0$, we can then approximate $\varsigma_\delta\phi$ via mollification. By taking appropriate subsequences, we obtain our aforementioned sequence $\phi_n \in C_c^\infty(\mathcal{M};\mathbb{R}^N)$. Notice that (1.3.4) holds for each ϕ_n.

But then

$$\int \langle \nabla(\phi_n - \phi), f\nabla u\rangle \leq c\,\|f\|_{L^\infty}\,\|\phi_n-\phi\|_{W^{1,2}}\,\|\nabla u\|_{L^2},$$

and

$$\int \langle \phi_n - \phi, fA(u)(\nabla u,\nabla u)\rangle \leq c\,\|f\|_{L^\infty}\,\|\phi_n-\phi\|_{L^\infty}\,\|\nabla u\|_{L^2}^2.$$

□

We may relax our definition of f-harmonic slightly: The variations that we consider need not smooth. We will need this later.

Lemma 1.3.3. *Let $u \in W^{1,2}(\mathcal{M};\mathcal{N})$ be weakly f-harmonic and let $\phi \in W_0^{1,2}\cap C^0(\mathcal{M};\mathbb{R}^N)$. Then $u_s := P\circ(u+s\phi)$ satisfies*

$$\left.\frac{d}{ds}E_f(u_s)\right|_{s=0} = 0.$$

Proof. Even with $\phi \in W_0^{1,2}\cap C^0$, the energy $E_f(P\circ(u+s\phi))$ is well defined and differentiable with respect to s. It follows by the same calculation used in (1.3.3) and by Lemma 1.3.2 that

$$\left.\frac{d}{ds}E_f(u_s)\right|_{s=0} = \int_{\mathcal{M}} \langle\nabla\phi, f\nabla u\rangle - \langle \phi, fA(u)(\nabla u,\nabla u)\rangle\, d\mathcal{M} = 0.$$

□

Definition 1.3.4 (Domain Variation). A *domain variation* is a map $\eta : \mathcal{M}\times(-\varepsilon,\varepsilon)\to\mathcal{M}$ for some small $\varepsilon>0$, which satisfies

$$\begin{cases} \eta(x,0) = x & \text{on } \mathcal{M} \\ \eta(x,s) = x & \text{on } \partial\mathcal{M}. \end{cases} \tag{1.3.5}$$

Lemma 1.3.5. *Let $u \in C^2(\mathcal{M};\mathcal{N})$ be an f-harmonic map. Let $\{\Omega_j\}$ be a finite partition of $\mathcal{M}$ such that $\partial\Omega_j \in C^\infty$ for each j. Suppose that the domain variation $\eta \in C^0(\mathcal{M} \times (-\varepsilon,\varepsilon);\mathcal{M})$ satisfies*

(i) $\eta \in C^2(\overline{\Omega}_j \times (-\varepsilon,\varepsilon);\mathcal{M})$ *for each* j*;*

(ii) $\frac{\partial\eta}{\partial s} \in C^0(\mathcal{M} \times (-\varepsilon,\varepsilon);T\mathcal{M})$*; and*

(iii) $\frac{\partial\eta}{\partial s} \in C^2(\overline{\Omega}_j \times (-\varepsilon,\varepsilon);T\mathcal{M})$ *for each* j*.*

Then

$$\frac{d}{ds}E_f(u \circ \eta)\bigg|_{s=0} = 0.$$

Proof. Notice that $u \circ \eta \in W^{1,2}(\mathcal{M},\mathcal{N})$. Define $\phi = \big\langle \nabla u^i, \frac{\partial\eta}{\partial s}(\cdot,0)\big\rangle \frac{\partial}{\partial u^i} \in C^0 \cap W_0^{1,2}(\mathcal{M};T\mathcal{N})$. Then

$$\frac{\partial}{\partial s}\big(u \circ \eta\big)\bigg|_{s=0} = \phi = dP_u(\phi) = \frac{\partial}{\partial s}\big(P \circ (u + s\phi)\big)\bigg|_{s=0}.$$

So

$$\begin{aligned}
\frac{d}{ds}E_f\big(u \circ \eta\big)\bigg|_{s=0} &= \sum_j \int_{\Omega_j} f \left\langle \frac{\partial}{\partial s}\nabla(u \circ \eta)\bigg|_{s=0}, \nabla u\right\rangle \, d\mathcal{M} \\
&= \sum_j \int_{\Omega_j} f \left\langle \frac{\partial}{\partial s}\nabla\big(P \circ (u + s\phi)\big)\bigg|_{s=0}, \nabla u\right\rangle \, d\mathcal{M} \\
&= \frac{d}{ds}E_f\big(P \circ (u + s\phi)\big)\bigg|_{s=0} = 0
\end{aligned}$$

by Lemma 1.3.3. □

1.4 Imagination

Now consider this question: How may we imagine an f-harmonic map? For (1-)harmonic maps, we have the following notion from Eells and Lemaire [EL78, page 1]: *"In physical terms, we imagine $\mathcal{M}$ made of 'rubber' and $\mathcal{N}$ of 'marble'; the map u constrains $\mathcal{M}$ to lie on $\mathcal{N}$. Then with each point $x \in \mathcal{M}$ we have a vector $\tau\big(u\big)(x) = \operatorname{div}\big(\nabla u(x)\big)$ at the point $u(x) \in \mathcal{N}$, representing the tension in the 'rubber' at that point. Thus u is harmonic if and only if u constrains $\mathcal{M}$ to lie on $\mathcal{N}$ in a position of elastic equilibrium."* Here the 'rubber' is some kind of 'special mathematical rubber' which is able to stretch infinitely without ever exceeding its elastic limit.

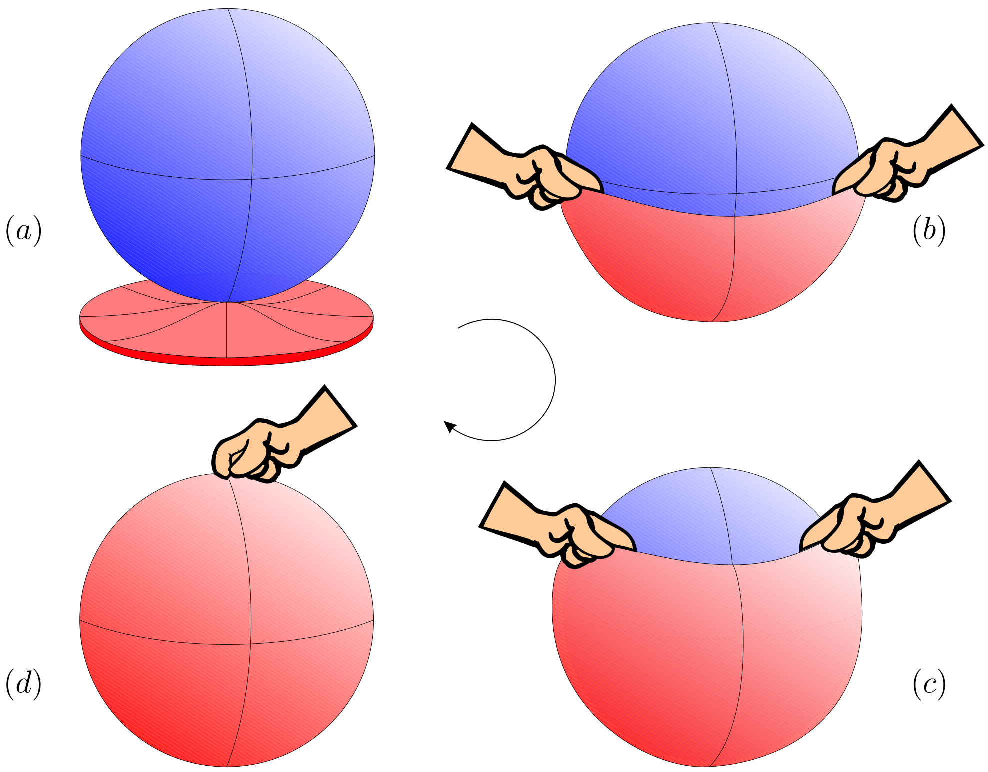

Figure 1.2: A rubber disc being wrapped around a marble 2-sphere, with the boundary of the disc 'held' in position at the 'north pole' (d). This is one way of thinking of an f-harmonic map $u : D \to S^2$ with the restriction that $\partial D \mapsto \{0\}$. Part (a) shows the rubber disc, not of uniform thickness, before stretching around S^2. This is a problem which is studied in chapter 4.

For f-harmonic maps, we may think along the same lines. Imagine here that the 'rubber', of which the domain $\mathcal{M}$ is made, is not of uniform thickness. Indeed, imagine that the thickness of the rubber at the point $x \in \mathcal{M}$ is of thickness $f(x)$. Then the thicker sections of the 'rubber' will have a stronger elasticity than the thinner sections. So analogously to before, u is f-harmonic if and only if the rubber lies on $\mathcal{N}$ in a position of elastic equilibrium. See figure 1.2 on page 8.

1.5 The f-Harmonic Map Heat Flow

In chapter 3, we study the L^2 gradient flow of the functional E_f – namely the following problem, which we call the *f-harmonic heat flow*:

$$\begin{cases} u_t - f(x)\Delta_{\mathcal{M}} u = f(x)A(u)(\nabla u, \nabla u) + \nabla f * \nabla u \\ u|_{t=0} = u_0 \\ u(\cdot, t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}. \end{cases} \tag{1.5.1}$$

Here $\partial\mathcal{M}$ may be empty or non-empty. Setting $f \equiv 1$, we obtain the harmonic map heat equation of Eells-Sampson [ES64].

Returning to our 'rubber and marble' notion, suppose that the rubber is not in elastic equilibrium. Then the rubber will be inclined to 'slide' along $\mathcal{N}$ in whichever direction reduces the tension (and hence the 'energy' stored in the rubber). This is exactly what we would see happen to the map u if it were to evolve under the f-harmonic map heat flow (1.5.1).

1.6 First and second variations of E_f, f-Tension and Jacobi Fields

1.6.1 Notation

Consider a map $u : (\mathcal{M}^m, g) \to (\mathcal{N}^n, h)$. For coordinates in $\mathcal{M}$ we will use Greek letters as indices, while coordinates in $\mathcal{N}$ will use Latin letters.

In the following calculations, f_α denotes $\frac{\partial f}{\partial x^\alpha}$, and $f_{\alpha\beta}$ denotes $\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}$. The notation $h_{ij,k}$ denotes $\frac{\partial h_{ij}}{\partial u^k}$. Moreover define

$$\partial_\alpha := \frac{\partial}{\partial x^\alpha} \quad , \quad e_i := \frac{\partial}{\partial u^i}. \tag{1.6.1}$$

For the purpose of reference, we note that

$$\begin{aligned} & v\ ,\ w \in \Gamma(u^*T\mathcal{N}), \\ & du\ ,\ u.\ ,\ d^\nabla w\ ,\ \nabla_{\frac{\partial}{\partial t}} du_t\big|_{t=0} \in \Gamma(T^*\mathcal{M}\otimes u^*(T\mathcal{N})) \\ \text{and}\quad & du. \in \Gamma(\Lambda^2 T^*\mathcal{M}\otimes u^*(T\mathcal{N})). \end{aligned} \tag{1.6.2}$$

1.6.2 First variation and f-Tension

Let u_t be an (admissible) variation of u. Set $w = \frac{du_t}{dt}\big|_{t=0}$. Then

$$(\delta E_f)(u)(w) := \frac{d}{dt}E_f(u_t)\bigg|_{t=0} = \int_{\mathcal{M}} f\left\langle du, \nabla_{\frac{\partial}{\partial t}} du_t\Big|_{t=0}\right\rangle d\mathcal{M}.$$

Note that the notation du_t denotes the differential on $T\mathcal{M}$ (not on $\mathrm{T}(\mathcal{M}\times\mathbb{R})$). Now

$$\nabla_{\frac{\partial}{\partial t}} du_t(\partial_\alpha)\Big|_{t=0} = \nabla_{\frac{\partial}{\partial t}} \frac{\partial u_t}{\partial x_\alpha}\bigg|_{t=0} = \nabla_{\frac{\partial}{\partial x_\alpha}} \frac{\partial u_t}{\partial t}\bigg|_{t=0} = \nabla_{\partial_\alpha} w \quad = d^\nabla w(\partial_\alpha)$$

so

$$(\delta E_f)(u)(w) = \int_{\mathcal{M}} f\left\langle du, d^\nabla w\right\rangle d\mathcal{M}.$$

Now [Lee97, p88], gives, for a k-tensor field ω and a $(k+1)$-tensor field η, the integration-by-parts formula

$$\int_{\mathcal{M}} \langle \nabla\omega, \eta\rangle\, d\mathcal{M} = -\int_{\mathcal{M}} \langle \omega, tr_g \nabla\eta\rangle\, d\mathcal{M} + \int_{\partial\mathcal{M}} \langle \omega\otimes N, \eta\rangle\, d\tilde{\mathcal{M}}.$$

(where $N \in T\mathcal{M}$ denotes the "outward normal" to $\partial\mathcal{M}$). Therefore

$$\begin{aligned} (\delta E_f)(u)(w) &= \int_{\mathcal{M}} \left\langle f du, d^\nabla w\right\rangle d\mathcal{M} \\ &= -\int_{\mathcal{M}} \langle tr_g \nabla f du, w\rangle\, d\mathcal{M} + \int_{\partial\mathcal{M}} \langle w\otimes N, f du\rangle\, d\tilde{\mathcal{M}} \\ &= -\int_{\mathcal{M}} \langle tr_g \nabla f du, w\rangle\, d\mathcal{M}. \end{aligned} \tag{1.6.3}$$

$\tau_f(u) := tr_g \nabla f du$ is the Euler-Lagrange operator associated with u.

Definition 1.6.1. We call $\tau_f(u)$ the *f-tension of u*.

1.6.3 Second variation and Jacobi Fields

We calculate now, the second variation of E_f. For the harmonic case, see [Smi75].

Let $u_{s,t}$ be a (compactly supported) variation of an f-harmonic map u. Let ∇^u denote the connection on $u^*T\mathcal{N}$. Set $v = \frac{du_{s,t}}{ds}\big|_{s,t=0}$ and $w = \frac{du_{s,t}}{dt}\big|_{s,t=0}$. Then the *Hessian* of u is

$$H_u(v,w) := \frac{\partial^2}{\partial s \partial t} E_f(u_{s,t})\Big|_{s,t=0}$$
$$= \int_{\mathcal{M}} f\Big[\Big\langle \nabla^u_{\frac{\partial}{\partial s}} du_{s,t}, \nabla^u_{\frac{\partial}{\partial t}} du_{s,t}\Big\rangle + \Big\langle \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} du_{s,t}, du_{s,t}\Big\rangle\Big] d\mathcal{M}\Big|_{s,t=0}. \tag{1.6.4}$$

Now

$$\begin{aligned}
\nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} du_{s,t}(\partial_\alpha)\Big|_{s,t=0} &= \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} \frac{\partial u_t}{\partial x_\alpha}\Big|_{s,t=0} = \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\partial_\alpha} \frac{\partial u_t}{\partial t}\Big|_{s,t=0} \\
&= \nabla^u_{\partial_\alpha} \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} u_{s,t} - R^u\left(\partial_\alpha, \frac{\partial}{\partial s}\right)\frac{\partial u_t}{\partial t}\Big|_{s,t=0} \\
&= d^\nabla \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} u_{s,t}\Big|_{s,t=0}(\partial_\alpha) - R^{\mathcal{N}}(du(\partial_\alpha), v)\, w.
\end{aligned} \tag{1.6.5}$$

Note also that

$$\begin{aligned}
\int_{\mathcal{M}} f\left\langle d^\nabla \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} u_{s,t}, du_{s,t}\right\rangle\Big|_{s,t=0} &= \int_{\mathcal{M}} \left\langle d^\nabla \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} u_{s,t}\Big|_{s,t=0}, f du\right\rangle \\
&= -\int_{\mathcal{M}} \left\langle \nabla^u_{\frac{\partial}{\partial s}} \nabla^u_{\frac{\partial}{\partial t}} u_{s,t}\Big|_{s,t=0}, tr_g \nabla f du\right\rangle = 0
\end{aligned} \tag{1.6.6}$$

as u is f-harmonic. It follows from (1.6.4) – (1.6.6) that

$$\begin{aligned}
H_u(v,w) &= \int_{\mathcal{M}} f\Big[\langle d^\nabla v, d^\nabla w\rangle - \langle R^{\mathcal{N}}(du, v)\, w, du\rangle\Big] d\mathcal{M} \\
&= -\int_{\mathcal{M}} \langle tr_g \nabla f d^\nabla v + f R^{\mathcal{N}}\big(v, du(\partial_\alpha)\big) du(\partial_\alpha), w\rangle\, d\mathcal{M} \\
&=: \int_{\mathcal{M}} \langle J_{f,u}(v), w\rangle\, d\mathcal{M},
\end{aligned} \tag{1.6.7}$$

where we have used the curvature identities

$$R(X,Y)Z = -R(Y,X)Z \quad \text{and} \quad \langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle.$$

Definition 1.6.2. The operator $J_{f,u} := -tr_g \nabla f d^{\nabla} - f R^{\mathcal{N}}\big(\cdot, du(\partial_\alpha)\big) du(\partial_\alpha)$ is called the *(E_f–)Jacobi Operator*. The vector fields $v \in \ker J_{f,u}$ are called the *(E_f–)Jacobi fields along u*.

1.7 Known facts about f-harmonic maps

We end this chapter by quoting two more results from the literature. From [EL78, §10.20: page 48] we state the result:

Lemma 1.7.1. *If $\mathcal{M}$ and $\mathcal{N}$ are compact Riemannian manifolds and* $\mathrm{Riem}^{\mathcal{N}} \leq 0$*, every homotopy class of maps from $\mathcal{M}$ to $\mathcal{N}$ contains an f-harmonic representative.*

Meanwhile, from A. Lichnerowicz [Lic70, page 367] we have the following:

Proposition 1.7.2 (Lichnerowicz). *Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be compact Riemannian manifolds. Suppose that there exists a function $f : \mathcal{M} \to (0, \infty)$ such that the tensor c defined by*

$$c_{AB} = R_{AB} - \nabla_A \nabla_B \log f$$

is positive. Suppose further that $(\mathcal{N}, h)$ has negative sectional curvature. Then:

(i) Every f-harmonic map $(\mathcal{M}, g) \to (\mathcal{N}, h)$ is totally geodesic;

(ii) If, in addition, the tensor c is positive definite at least one point in $\mathcal{M}$, then every f-harmonic map must be a constant map;

(iii) Conversely, if the sectional curvature of $(\mathcal{N}, h)$ is strictly negative at every point in $\mathcal{N}$, every f-harmonic map $\mathcal{M} \to \mathcal{N}$ is either constant, or a map of rank 1 which maps $\mathcal{M}$ onto a closed geodesic of $(\mathcal{N}, h)$.

Chapter 2

Properties of f-harmonic maps

2.1 Puncture Repair Kit

2.1.1 Comment

In this section we show that, a smooth f-harmonic map with finite energy, defined on the punctured disc $D \setminus \{0\}$ extends smoothly to the whole disc. We will need this result in chapter 3 where we study the f-harmonic map heat flow.

The equivalent result for harmonic maps is due to Sacks and Uhlenbeck [SU81, Theorem 3.6]. Their proof used a nice property of the Hopf Differential ϕdz^2, where $\phi(u) = |u_x|^2 - |u_y|^2 - 2i \langle u_x, u_y \rangle$, to show that: *Let $u : D \setminus \{0\} \to \mathcal{N} \subset \mathbb{R}^n$ be a smooth harmonic map such that $E_1(u) < \infty$. Then $\int_0^{2\pi} |u_\theta(z)|^2 \, d\theta = r^2 \int_0^{2\pi} |u_r(z)|^2 d\theta$.* In our case, we may only hope for something like:

Lemma 2.1.1. *Let $u : D \setminus \{0\} \to \mathcal{N} \subset \mathbb{R}^n$ be a smooth f-harmonic map such that $E_f(u) < \infty$. Then, for $0 < \rho \leq \frac{1}{2}$, there holds*

$$\int_{D_\rho} |\nabla u|^2 \, dx \leq 2 \int_{D_\rho} \frac{1}{r^2} |u_\theta|^2 \, dx + C_1 \rho \, \|\nabla f\|_{L^\infty} E_f(u; D) \tag{2.1.1}$$

where $C_1 = C_1(f)$.

Here $D_\rho := B_\rho(0) \subset \mathbb{R}^2$. In the case where $\rho = 1$ the subscript is dropped.

Remark. The reader should not be concerned that this result gives an estimate on the $L^2(D_\rho)$ norm of the gradient of u, whereas Sacks and Uhlenbeck considered the integral over $[0, 2\pi]$ of $|u_r|^2$. This difference would not prevent this result being used in the same way.

However for our purposes with f-harmonic maps it is more elegant to prove the Puncture Repair Theorem (Theorem 2.1.5) without using Lemma 2.1.1, so a proof of this lemma will not be given here.

Instead we will provide a proof relying on the following theorem of R. Moser [Mos03, Theorem 1] – who has studied the *Approximated Harmonic Map Equation* (2.1.2).

Theorem 2.1.2 (Moser). *Let $\Omega \subset \mathbb{R}^n$, ($n \geq 2$) be an open domain. Suppose that $u \in W^{1,2}(\Omega; \mathcal{N})$ is a solution of*

$$\Delta u + A(u)(\nabla u, \nabla u) = F \tag{2.1.2}$$

with $F \in L^p(\Omega, \mathbb{R}^N)$ for $p > \frac{n}{2}$. There exist numbers $\bar{\varepsilon} > 0$ and $\alpha \in (0,1]$, depending on p, n and N, such that if u satisfies

$$\|\nabla u\|_{M^{2,n-2}(\Omega)} \leq \varepsilon$$

with $\varepsilon \leq \bar{\varepsilon}$, then $u \in C^{0,\alpha}_{loc}(\Omega; \mathcal{N})$. Moreover,

$$[u]_{C^{0,\alpha}(\Omega')} \leq C \left(\varepsilon + \|F\|_{L^p(\Omega)}\right)$$

for any $\Omega' \subset\subset \Omega$, where the constant C depends on p, n, N, Ω and Ω'.

Here $\|\cdot\|_{M^{q,\lambda}(\Omega)}$ is the norm of the Morrey space $M^{q,\lambda}(\Omega)$, i.e.

$$\|F\|_{M^{q,\lambda}(\Omega)} = \sup_{B_r(x_0) \subset \Omega} \left(r^{-\lambda} \int_{B_r(x_0)} |F|^q \, dx \right)^{\frac{1}{q}},$$

for $\lambda \geq 0$ and $1 \leq q < \infty$.

Remark. f-harmonic maps are *approximate harmonic maps* with $F := -\frac{1}{f}\nabla f * \nabla u \in L^2(\mathcal{M}; \mathcal{N})$.

2.1.2 Tool Kit

Lemma 2.1.3. *Suppose that $u : D \setminus \{0\} \to \mathcal{N}$ is a smooth f-harmonic map with finite energy. Then u is a weakly f-harmonic map $D \to \mathcal{N}$.*

Proof. Notice first that $u \in W^{1,2}(D; \mathcal{N})$. Since u is f-harmonic, we have that

$$-\Delta u = A(u)(\nabla u, \nabla u) + \frac{1}{f}\nabla f * \nabla u \tag{2.1.3}$$

on $D \setminus \{0\}$.

Now let $\psi \in C_c^\infty(D;\mathbb{R}^N)$. There exist $\psi_i \in C_c^\infty(D;\mathbb{R}^N)$ such that $\psi_i \to \psi$ in $W^{1,2} \cap L^\infty$ and such that each ψ_i is constant in some neighbourhood of 0.

Then for small ε, define $\eta_i^\varepsilon \in C_c^\infty(D \setminus \{0\};\mathbb{R}^N)$ by

$$\eta_i^\varepsilon(x) = \begin{cases} \psi_i(x) - \xi_i^\varepsilon(x) & \text{if } |x| \leq \varepsilon \\ \psi_i(x) & \text{if } |x| > \varepsilon \end{cases}$$

where $\xi_i^\varepsilon(x) := \psi_i(\frac{x}{\varepsilon})$. So by (2.1.3), we see that

$$\int_{D\setminus\{0\}} \nabla u \cdot \nabla \eta_i^\varepsilon = \int_{D\setminus\{0\}} \eta_i^\varepsilon \Big[A(u)(\nabla u, \nabla u) + \frac{1}{f} \nabla f * \nabla u \Big],$$

and it follows that

$$\int_D \nabla u \cdot \nabla \eta_i^\varepsilon = \int_D \eta_i^\varepsilon \Big[A(u)(\nabla u, \nabla u) + \frac{1}{f} \nabla f * \nabla u \Big].$$

We will now take the limit $\varepsilon \to 0$ to show that this equality also holds with ψ_i in place of η_i^ε. Indeed

$$\begin{aligned} \Big| \int_D \nabla u \cdot \nabla \psi_i - \nabla u \cdot \nabla \eta_i^\varepsilon \Big| &= \Big| \int_{D_\varepsilon} \nabla u \cdot \nabla \xi_i^\varepsilon \Big| \\ &\leq \Big(\int_{D_\varepsilon} |\nabla u|^2 \Big)^{\frac{1}{2}} \Big(\int_{D_\varepsilon} \big|\nabla \xi_i^\varepsilon\big|^2 \Big)^{\frac{1}{2}} \\ &= \Big(\int_{D_\varepsilon} |\nabla u|^2 \Big)^{\frac{1}{2}} \Big(\int_{D} \big|\nabla \psi_i\big|^2 \Big)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $\varepsilon \to 0$. Moreover

$$\begin{aligned} \Big| \int_D \psi_i A(u)(\nabla u, \nabla u) - \eta_i^\varepsilon A(u)(\nabla u, \nabla u) \Big| &= \Big| \int_{D_\varepsilon} \xi_i^\varepsilon A(u)(\nabla u, \nabla u) \Big| \\ &\leq c(\mathcal{N}, \psi_i) \int_{D_\varepsilon} |\nabla u|^2 \to 0 \end{aligned}$$

and

$$\begin{aligned} \Big| \int_D \psi_i \frac{1}{f} \nabla f * \nabla u - \int_D \eta_i^\varepsilon \frac{1}{f} \nabla f * \nabla u \Big| &= \Big| \int_{D_\varepsilon} \xi_i^\varepsilon \frac{1}{f} \nabla f * \nabla u \Big| \\ &\leq c(f, \psi_i) \int_{D_\varepsilon} |\nabla u| \to 0 \end{aligned}$$

as $\varepsilon \to 0$. Therefore we must have that

$$\int_D \nabla u \cdot \nabla \psi_i = \int_D \psi_i \Big[A(u)(\nabla u, \nabla u) + \frac{1}{f}\nabla f * \nabla u\Big].$$

Finally, we let $i \to \infty$, and use the $W^{1,2} \cap L^\infty$–convergence of $\psi_i \to \psi$ to see that u extends to a weakly f-harmonic map on D.

□

The next result is based on a result of Schoen [Sch84, Lemma 3.1].

Lemma 2.1.4. *Suppose that* $u \in C^{0,\alpha}(D;\mathcal{N}) \cap W^{1,2}(D;\mathcal{N})$ *for some* $\alpha \in (0,1)$, *is a weakly f-harmonic map. Then* $u \in C^{0,1}(D_{\frac{1}{2}};\mathcal{N}) = W^{1,\infty}(D_{\frac{1}{2}};\mathcal{N})$.

Proof.
Let $y \in D_{\frac{1}{2}}$ and $\sigma \in (0, \frac{1}{4})$. Suppose that $v \in W^{1,2}(B_\sigma(y);\mathbb{R}^N) \cap C^0(B_\sigma(y);\mathbb{R}^N)$ solves

$$\begin{cases} \Delta v = 0 & \text{on } B_\sigma(y) \\ v = u & \text{on } \partial B_\sigma(y). \end{cases} \tag{2.1.4}$$

Since $|\nabla v|^2$ is sub-harmonic on $B_\sigma(y)$, it follows that

$$\frac{d}{dr}\left(⨍_{B_r(y)} |\nabla v|^2\right) \geq 0\ , \qquad r \in (0,\sigma). \tag{2.1.5}$$

By (2.1.4), we also have that

$$\int_{B_\sigma(y)} \nabla(u-v)\cdot\nabla v = 0. \tag{2.1.6}$$

Moreover, for any $\phi \in W_0^{1,2}(B_\sigma(y);\mathbb{R}^N) \cap C^0(B_\sigma(y);\mathbb{R}^N)$

$$\int_{B_\sigma(y)} \nabla u \cdot \nabla \phi = \int_{B_\sigma(y)} \phi\Big[A(u)(\nabla u, \nabla u) + \frac{1}{f}\nabla f * \nabla u\Big],$$

by Lemma 1.3.2, so it follows (by setting $\phi = u - v$) that

$$\left|\int_{B_\sigma(y)} \nabla(u-v)\cdot\nabla u\right| \leq C \sup_{B_\sigma(y)} |u-v| \int_{B_\sigma(y)} \Big[|\nabla u|^2 + |\nabla u|\Big]. \tag{2.1.7}$$

It follows from (2.1.6) and (2.1.7) that

$$\begin{aligned}\int_{B_\sigma(y)} |\nabla(u-v)|^2 &= \int_{B_\sigma(y)} \nabla(u-v)\cdot\nabla u\\ &\leq C \sup_{B_\sigma(y)} |u-v| \int_{B_\sigma(y)} \Big[|\nabla u|^2 + |\nabla u|\Big].\end{aligned}$$

We calculate

$$\begin{aligned}\int_{B_{\frac{\sigma}{2}}(y)} |\nabla u|^2 &= -\int_{B_{\frac{\sigma}{2}}(y)} |\nabla u|^2 + 2\int_{B_{\frac{\sigma}{2}}(y)} \nabla u\cdot\nabla v + 2\int_{B_{\frac{\sigma}{2}}(y)} \nabla(u-v)\cdot\nabla u\\ &\leq \int_{B_{\frac{\sigma}{2}}(y)} |\nabla v|^2 + 2\left(\int_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla(u-v)|^2\right)^{\frac{1}{2}}\end{aligned}$$

(where we have used the identity $2ab \leq a^2+b^2$ on the $\nabla u\cdot\nabla v$ term)

$$\leq \int_{B_{\frac{\sigma}{2}}(y)} |\nabla v|^2 + C\sup_{B_\sigma(y)} |u-v|^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{1}{2}}\left(\int_{B_\sigma(y)} \Big[|\nabla u|^2+|\nabla u|\Big]\right)^{\frac{1}{2}}$$

(by (2.1.7))

$$\begin{aligned}&\leq \int_{B_{\frac{\sigma}{2}}(y)} |\nabla v|^2 + C\sup_{B_\sigma(y)} |u-v|^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{1}{2}}\\ &\quad + C\sup_{B_\sigma(y)} |u-v|^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla u|\right)^{\frac{1}{2}}\end{aligned}$$

(where we have used the identity $\sqrt{a^2+b^2} \leq a+b$ for $a,b\geq 0$)

$$\begin{aligned}&\leq \int_{B_{\frac{\sigma}{2}}(y)} |\nabla v|^2 + C\sup_{B_\sigma(y)} |u-v|^{\frac{1}{2}}\int_{B_\sigma(y)} |\nabla u|^2\\ &\quad + C\sigma^{\frac{1}{2}}\sup_{B_\sigma(y)} |u-v|^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{1}{2}}\left(\int_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{1}{4}}.\end{aligned}$$

Now, since u is Hölder continuous and v satisfies the maximum principle, it follows that $\sup_{B_\sigma(y)} |u-v| \leq (2\sigma)^\alpha$. Therefore

$$⨍_{B_{\frac{\sigma}{2}}(y)} |\nabla u|^2 \leq ⨍_{B_{\frac{\sigma}{2}}(y)} |\nabla v|^2 + C\sigma^{\frac{\alpha}{2}} ⨍_{B_\sigma(y)} |\nabla u|^2 + C\sigma^{\frac{\alpha}{2}}\left(⨍_{B_\sigma(y)} |\nabla u|^2\right)^{\frac{3}{4}}.$$

However, by (2.1.5), and Dirichlet's Principle

$$\fint_{B_{\frac{\sigma}{2}}(y)} |\nabla v|^2 \leq \fint_{B_\sigma(y)} |\nabla v|^2 \leq \fint_{B_\sigma(y)} |\nabla u|^2,$$

so

$$\begin{aligned}\fint_{B_{\frac{\sigma}{2}}(y)} |\nabla u|^2 &\leq (1 + C\sigma^{\frac{\alpha}{2}}) \fint_{B_\sigma(y)} |\nabla u|^2 + C\sigma^{\frac{\alpha}{2}} \left(\fint_{B_\sigma(y)} |\nabla u|^2 \right)^{\frac{3}{4}} \\ &\leq (1 + C\sigma^{\frac{\alpha}{2}}) \max\left\{1,\ \fint_{B_\sigma(y)} |\nabla u|^2 \right\}.\end{aligned}$$

We iterate, starting at $\sigma = \frac{1}{4}$, to see that

$$\fint_{B_{\frac{1}{4}\left(\frac{1}{2}\right)^m}(y)} |\nabla u|^2 \leq \prod_{j=0}^{m-1} \left(1 + C \left[\frac{1}{4} \left(\frac{1}{2} \right)^j \right]^{\frac{\alpha}{2}} \right) \cdot \max\left\{1,\ \fint_{B_{\frac{1}{4}}(y)} |\nabla u|^2 \right\}.$$

Therefore

$$\fint_{B_{\left(\frac{1}{2}\right)^{m+2}}(y)} |\nabla u|^2 \leq \prod_{j=0}^{\infty} \left(1 + C \left[\frac{1}{2} \right]^{(j+2)\frac{\alpha}{2}} \right) \cdot \max\left\{1,\ \fint_{B_{\frac{1}{4}}(y)} |\nabla u|^2 \right\}.$$

But the right hand side is finite and independent of m, so

$$\fint_{B_{\frac{1}{2^k}}(y)} |\nabla u|^2 \leq K \tag{2.1.8}$$

for all $k \in \mathbb{N} \setminus \{1, 2\}$ and $y \in D_{\frac{1}{2}}$. It follows that $|\nabla u(y)|^2 \leq K$ for almost every $y \in D_{\frac{1}{2}}$. Hence $u \in W^{1,\infty}(D_{\frac{1}{2}})$. □

2.1.3 Puncture Repair

Finally we prove the "Removal of Singularity", or "Puncture Repair" Theorem.

Theorem 2.1.5 (Puncture Repair). *Suppose that $u : D \setminus \{0\} \to \mathcal{N}$ is a smooth f-harmonic map with finite energy. Then u extends to a smooth f-harmonic map $u : D \to \mathcal{N}$.*

Remark. Fundamentally, this result is true because the set $\{0\}$ has zero capacity – see e.g. [EP84].

Proof of Theorem 2.1.5.

We wish to use the aforementioned result of Moser [Mos03, Theorem 1] (reproduced here as Theorem 2.1.2). Notice that $F := -\frac{1}{f}\nabla f * \nabla u \in L^2(D;\mathcal{N})$ so we may apply this theorem. Moreover $\|\nabla u\|_{M^{2,0}(D)} \leq \|\nabla u\|_{L^2(D)}$, and we may assume (by scaling in the domain) that $\|\nabla u\|_{L^2(D)} \leq \varepsilon \leq \bar{\varepsilon}$. Therefore there exists $\alpha \in (0,1]$ such that $u \in C^{0,\alpha}_{loc}(D)$. It follows that $u \in C^{0,\alpha}(\overline{D}_\theta)$, $\theta \in (0,1)$.

All that remains is to "bootstrap". By Lemma 2.1.4, we also have that $u \in W^{1,\infty}(D_{\frac{1}{2}})$. It follows that

$$|\Delta u| \leq C\left(|\nabla u|^2 + |\nabla u|\right) < \infty,$$

so $\Delta u \in L^\infty$. By Calderon-Zygmund theory, (see e.g. [GT70, Theorem 9.11]),

$$\begin{aligned}\|u\|_{W^{2,p}(D_{\frac{1}{2}\theta})} &\leq C\left(\|u\|_{L^p(D_{\frac{1}{2}})} + \|\Delta u\|_{L^p(D_{\frac{1}{2}})}\right)\\ &\leq C\left(\|u\|_{L^\infty(D_{\frac{1}{2}})} + \|\Delta u\|_{L^\infty(D_{\frac{1}{2}})}\right) < \infty,\end{aligned}$$

for any $2 < p < \infty$. Then by Morrey's Inequality

$$\|Du\|_{C^{0,\alpha}(D_{\frac{1}{2}\theta})} \leq C\,\|Du\|_{W^{1,p}(D_{\frac{1}{2}\theta})}$$

where $\alpha = 1 - \frac{2}{p}$. Therefore $u \in C^{1,\alpha}(D_{\frac{1}{2}\theta})$ for all $\theta \in (0,1)$ and for all $\alpha \in (0,1)$.

It follows by Schauder regularity theory that $u \in C^\infty(D_{\frac{1}{2}\theta})$ for all $\theta \in (0,1)$.

□

2.2 An application of the Implicit Function Theorem to f-harmonic maps

We suppose in this section that $\mathcal{M}$ is a compact surface *without* boundary.

2.2.1 Comment

Consider the following question: If we have an f-harmonic map $u : \mathcal{M} \to \mathcal{N}$, and then we vary the function f by a small amount (to f_1 say), is it possible to distort the map u by a small amount so as to create an f_1-harmonic map?

This is clearly not always possible as the following example shows: Consider $f \equiv 1$ and let $u : S^2 \to S^2$ be the identity map. Here the two spheres S^2 are given the usual metric. The map u is f-harmonic. Now consider the new function $f_1 : S^2 \to [1-\varepsilon, 1+\varepsilon]$ with $f_1(\text{north pole}) = 1+\varepsilon$, $f_1(\text{south pole}) = 1-\varepsilon$ and f_1 'defined suitably' in $(1-\varepsilon, 1+\varepsilon)$ elsewhere. For sufficiently small $\varepsilon > 0$, this function can be considered to be "close" to f in any appropriate sense. However any map $u_1 : S^2 \to S^2$ "close" to u will be inclined to flow[1] to the south pole, so can not be f_1-harmonic.

However, we are able to offer a result (Proposition 2.2.2), with a particular hypothesis on the Jacobi operator (Definition 1.6.2 on page 12), which gives a positive answer to the considered question. Our result makes use of the Implicit Function Theorem, so we state this now in the form that we shall use.

Theorem 2.2.1 (Implicit Function Theorem). *Let X, Y, Z be Banach spaces. Let $U \times V$ be an open subset of $X \times Y$. Suppose that $G : U \times V \to Z$ is continuous and has the property that $d_2 G$ exists and is continuous at each point of $U \times V$. Assume that the point $(x, y) \in U \times V$ has the property that $G(x, y) = 0$ and that $d_2 G(x, y)$ is invertible.*

Then there are open balls $R = B^X_\delta(x)$ and $S = B^Y_\varepsilon(y)$ such that, for each $r \in R$, there is a unique $s \in S$ satisfying $G(r, s) = 0$. The function $F : R \to S$, thereby uniquely determined near x by the condition $F(r) = s$, is continuous.

We now present our application. Here, as always, $f \in C^\infty(\mathcal{M}; (0,\infty))$ and $\mathcal{N} \subset \mathbb{R}^N$.

Proposition 2.2.2. *Let $\mathcal{M}$ be a compact surface without boundary. Suppose that $u \in C^\infty\big((\mathcal{M}, g); (\mathcal{N}, h)\big)$ is a f-harmonic map. Suppose further that there are no Jacobi Fields along u. Let $\alpha \in (0,1)$. There exist $\delta > 0$ and $\varepsilon > 0$ such that; if $f_1 \in C^\infty(\mathcal{M}; (0,\infty))$ and $\|f_1 - f\|_{C^{1,\alpha}} < \delta$ then there exists a unique f_1-harmonic map $u_1 \in C^\infty(\mathcal{M}; \mathcal{N})$ such that $\|u_1 - u\|_{C^{2,\alpha}} < \varepsilon$.*

Remark. The hypothesis that *there are no Jacobi Fields along* u implies immediately that u must be non-constant.

Remark. In spirit, this proposition is similar to the following result of Eells-Lemaire [EL81, Theorem 3.1] which is also an application of the Implicit Function Theorem. Here, the notation $\mu^{r,\alpha}(\mathcal{M})$ denotes the space of all Riemannian metrics on $\mathcal{M}$ which have α-Hölder continuous derivatives up to order r.

[1] see chapter 3 for a description of f-harmonic map heat flow

Theorem 2.2.3 (Eells-Lemaire). *Suppose that* $u_0 : (\mathcal{M}, g_0) \to (\mathcal{N}, h_0)$ *is a smooth harmonic map between smooth compact Riemannian manifolds, such that* $\ker J_{1,u_0} = \{0\}$. *Then for* $1 \leq k < \infty$ *and* $r < \infty$, *there is a neighbourhood* $\mathcal{V}$ *of* (g_0, h_0) *in* $\mu^{r+1,\alpha}(\mathcal{M}) \times \mu^{r+k+1,\alpha}(\mathcal{N})$ *and a unique* C^k*-map* $\sigma : \mathcal{V} \to C^{r+2,\alpha}(\mathcal{M}, \mathcal{N})$ *such that* $\sigma(g_0, h_0) = u_0$ *and* $\sigma(g, h)$ *is a harmonic map* $u : (\mathcal{M}, g) \to (\mathcal{N}, h)$.

Remark. Eells-Lemaire proved their result considering a *smooth manifold modelled on a Banach space.* Here however, we will define a certain "tubular neighbourhood" of $\mathcal{N}$, so that we may apply the simpler form of the Implicit Function Theorem.

2.2.2 Construction of the tubular neighbourhood

We wish to apply the *Implicit Function Theorem* to the f-tension, τ_f (page 10). However we may not do this directly as $C^\infty(\mathcal{M}; (0, \infty))$ and $C^\infty(\mathcal{M}; \mathcal{N})$ are not Banach spaces. As f is smooth and $\mathcal{M}$ is compact, we have that f is also a member of the Banach space $C^{1,\alpha}\big(\mathcal{M}; \mathbb{R}\big)$ for any $\alpha \in (0, 1)$.

For u, we must construct a Banach space of maps from $\mathcal{M}$ into $\mathbb{R}^N$. The reader is directed to [Hél02, proof of Lemma 4.1.2] (or to [Ham75, page 108]) for greater detail than will be given here. As in chapter 1, we define a tubular neighbourhood of $\mathcal{N}$ by

$$V_\rho \mathcal{N} := \{z \in \mathbb{R}^N : d(z, \mathcal{N}) < \rho\} \subset \mathbb{R}^N. \tag{2.2.1}$$

Also as previously, we let $P : V_\rho \mathcal{N} \to \mathcal{N}$ denote "nearest point" projection. Consider further the differential $dP_z : T_z \mathbb{R}^N \simeq \mathbb{R}^N \to T_{P(z)} \mathcal{N}$. Now define the "cross-section" at $y \in \mathcal{N}$ (see figure 2.1), to be

$$U_y := \{z \in V_\rho \mathcal{N} : P(z) = y\}.$$

For $z \in V_\rho \mathcal{N}$, we then define

$$Q_z : T_z \mathbb{R}^N \simeq \mathbb{R}^N \to T_z U_{P(z)}$$

to be the function which maps each $\xi \in T_z \mathbb{R}^N$ to its "vertical" component $Q_x(\xi)$. That is $Q_z(\xi) = \xi - dP_z(\xi)$.

Let $\gamma > 0$ be some small number. We define the metric h_1 on $V_\rho \mathcal{N}$ by

$$\begin{aligned} h_1(z)(\xi, \eta) := &\Big(1 + \big|z - P(z)\big|^2_{\mathbb{R}^N}\Big) h\big(P(z)\big) \Big(dP_z(\xi), dP_z(\eta)\Big) \\ &+ \gamma \left\langle Q_z(\xi), Q_z(\eta) \right\rangle_{\mathbb{R}^N} \end{aligned} \tag{2.2.2}$$

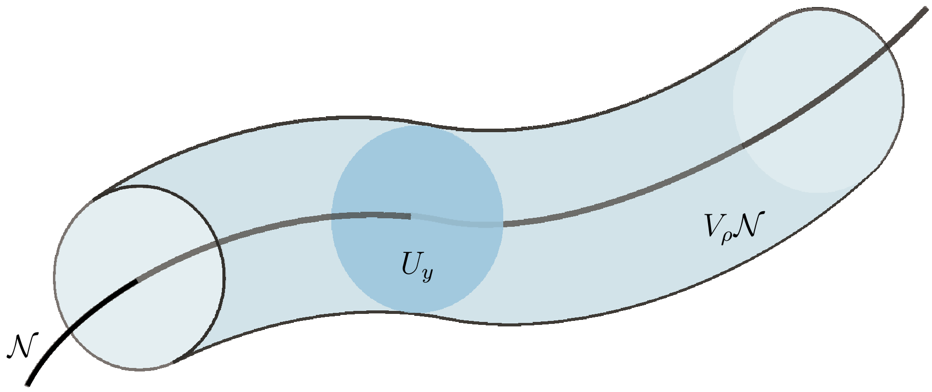

Figure 2.1: The tubular neighbourhood $V_\rho\mathcal{N}$.

for all $z \in V_\rho\mathcal{N}$ and $\xi, \eta \in T_z\mathbb{R}^N \simeq \mathbb{R}^N$. Intuitively, this means that "all distances increase" as we move away from $\mathcal{N}$. Extend h_1 to the rest of $\mathbb{R}^N$ in some arbitrary smooth way.

Remark. Our strategy is to consider u as an element of $C^{2,\alpha}\big((\mathcal{M}, g); (\mathbb{R}^N, h_1)\big)$. We may then apply the Implicit Function Theorem to see that for an f_1 'near' f, there is a u_1 'near' to u. Finally, we will use the following lemma to see u_1 as a map $\mathcal{M} \to \mathcal{N}$.

Lemma 2.2.4. *Let $u : (\mathcal{M}, g) \to (\mathbb{R}^N, h_1)$ be a smooth non-constant f-harmonic map. If $u(\mathcal{M}) \subset V_\rho\mathcal{N}$ then $u(\mathcal{M}) \subset \mathcal{N}$.*

Proof.

By constructing a variation of u, we show that for u to be an f-harmonic map, the image of u must be contained in $\mathcal{N}$. Define $u_{(\cdot)}(\cdot) : (-\varepsilon, \varepsilon) \times (\mathcal{M}, g) \to (\mathbb{R}^N, h_1)$ by $u_t(x) = u(x) + t\big(P(u(x)) - u(x)\big)$. Then we see from the definition of h_1 (2.2.2) that

$$\begin{aligned}
E_f(u_t) &= \frac{1}{2}\int_{\mathcal{M}} f\big|(1-t)\nabla u + t\nabla\big(P(u)\big)\big|_{h_1}^2 \; d\mathcal{M} \\
&= \frac{1}{2}\int_{\mathcal{M}} f\big|(1-t)\nabla u + t dP_u(\nabla u)\big|_{h_1}^2 \; d\mathcal{M} \\
&= \frac{1}{2}\int_{\mathcal{M}} f\Big\{\big(1+\big|u_t - P(u_t)\big|_{\mathbb{R}^N}^2\big)\big|(1-t)dP_u(\nabla u) + t dP_u(dP_u(\nabla u))\big|_h^2 \\
&\qquad + \gamma\big|(1-t)Q_u(\nabla u)\big|_{\mathbb{R}^N}^2\Big\} \; d\mathcal{M} \\
&= \frac{1}{2}\int_{\mathcal{M}} f\Big\{\big(1+(1-t)^2\big|u - P(u)\big|_{\mathbb{R}^N}^2\big)\big|dP_u(\nabla u)\big|_h^2 \\
&\qquad + \gamma(1-t)^2\big|Q_u(\nabla u)\big|_{\mathbb{R}^N}^2\Big\} \; d\mathcal{M}.
\end{aligned}$$

But because u is f-harmonic,

$$\begin{aligned}
0 &= \frac{d}{dt}E_f(u_t)\bigg|_{t=0} \\
&= -\int_{\mathcal{M}} f\big|u-P(u)\big|_{\mathbb{R}^N}^2\big|dP_u(\nabla u)\big|_h^2 \; d\mathcal{M} - \int_{\mathcal{M}} f\gamma\big|Q_u(\nabla u)\big|_{\mathbb{R}^N}^2.
\end{aligned}$$

Therefore, as u is non-constant, we must have that $u - P(u) = 0$ almost everywhere. Finally, u is smooth so $u(\mathcal{M}) \subset \mathcal{N}$. □

There is one more result we need before we may prove Proposition 2.2.2. Intuitively, as we move away from $\mathcal{N}$ "all distances increase" (due to the definition of h_1) . This implies that any vector field which is not contained in $T\mathcal{N}$, cannot be a Jacobi Field. So even though we are now working in $(\mathbb{R}^N, h_1)$ instead of $(\mathcal{N}, h)$, there are no extra Jacobi Fields along u. Precisely:

Lemma 2.2.5. *Let $u : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be an f-harmonic map. Denote the composition of u with the embedding $(\mathcal{N}, h) \subset (\mathbb{R}^N, h_1)$ by $\tilde{u}$. If there are no Jacobi Fields along u, then there are no Jacobi Fields along $\tilde{u}$.*

Proof. Suppose that $X \in \Gamma(\tilde{u}^*\mathbb{R}^N)$ $(X \not\equiv 0)$ is some Jacobi Field along $\tilde{u}$. Write $X = v + w$ where $v \in T_u\mathcal{N}$ and $w \in T_{\tilde{u}}U_{P(\tilde{u})}$. Notice first that J is linear, so

$$0 \equiv J_{f,\tilde{u}}^{(\mathbb{R}^N,h_1)}(v+w) = J_{f,\tilde{u}}^{(\mathbb{R}^N,h_1)}(v) + J_{f,\tilde{u}}^{(\mathbb{R}^N,h_1)}(w). \tag{2.2.3}$$

Note that $\mathcal{N}$ is totally geodesic in $(\mathbb{R}^N, h_1)$. Due to the way $\mathcal{N}$ is embedded in $\mathbb{R}^N$, the connection on $(\mathcal{N}, h)$ is just the tangential connection (see e.g. [Lee97, page 66]) of the connection on $(\mathbb{R}^N, h_1)$. It follows from the definition of the curvature tensor R that

$$\begin{aligned}J^{(\mathbb{R}^N,h_1)}_{f,\tilde{u}}(v) &= -tr_g\nabla f d^\nabla v - fR^{(\mathbb{R}^N,h_1)}(v, d\tilde{u}(\partial_\alpha))d\tilde{u}(\partial_\alpha)\\ &= -tr_g\nabla f d^\nabla v - fR^{(\mathcal{N},h)}(v, du(\partial_\alpha))du(\partial_\alpha)\\ &= J^{(\mathcal{N},h)}_{f,u}(v) \in T_u\mathcal{N}.\end{aligned}$$

Next let $\tilde{u}_{s,t} := \tilde{u} + (s+t)w$. Thus

$$\begin{aligned}&\int_{\mathcal{M}} \left\langle J^{(\mathbb{R}^N,h_1)}_{f,\tilde{u}}(w), w\right\rangle \, d\mathcal{M} = H_{\tilde{u}}(w,w) = \left.\frac{\partial^2 E_f(\tilde{u}_{s,t})}{\partial s \partial t}\right|_{s=t=0}\\ &\quad = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\int_{\mathcal{M}} f\big|d\tilde{u}_{s,t}(\cdot)\big|^2_{h_1}\, d\mathcal{M}\Bigg|_{s=t=0}\\ &\quad = \frac{1}{2}\int_{\mathcal{M}} f\,\frac{\partial^2}{\partial s\partial t}\left[\left(1+\big|\tilde{u}_{s,t}-\tilde{u}\big|^2_{\mathbb{R}^N}\right)\big|d\tilde{u}(\cdot)\big|^2_h + \gamma\big|Q_{\tilde{u}}(d\tilde{u}_{s,t}(\cdot))\big|^2_{\mathbb{R}^N}\right]\Bigg|_{s=t=0} d\mathcal{M}.\end{aligned}$$

As $\big|Q_{\tilde{u}}(d\tilde{u}_{s,t}(\cdot))\big|^2_{\mathbb{R}^N}$ is zero at $(s,t) = (0,0)$ and strictly positive for other (s,t), it follows that $\frac{\partial^2}{\partial s\partial t}\big|Q_{\tilde{u}}(d\tilde{u}_{s,t}(\cdot))\big|^2\Big|_{s=t=0} \geq 0$. Moreover $\frac{\partial^2}{\partial s\partial t}\big|\tilde{u}_{s,t}-\tilde{u}\big|^2\Big|_{s=t=0} = 2|w|^2 > 0$ (if $w \neq 0$). Therefore

$$\int_{\mathcal{M}} \left\langle J^{(\mathbb{R}^N,h_1)}_{f,\tilde{u}}(w), w\right\rangle \, d\mathcal{M} > 0 \qquad (\text{if } w \not\equiv 0)$$

and so either, the $T_{\tilde{u}}U_{P(\tilde{u})}$ component of $J^{(\mathbb{R}^N,h_1)}_{f,\tilde{u}}(w)$ is not identically zero, or $w \equiv 0$.

Finally we look again at (2.2.3). That $J^{(\mathbb{R}^N,h_1)}_{f,\tilde{u}}(X) \equiv 0$ means, in particular, that $J^{(\mathbb{R}^N,h_1)}_{f,\tilde{u}}(X)$ must have zero $T_{\tilde{u}}U_{P(\tilde{u})}$ component. This can only be true if $w \equiv 0$. But this implies that $v \not\equiv 0$ and that $J^{(\mathcal{N},h)}_{f,u}(v) = 0$ – i.e. the existence of a Jacobi Field along u. □

2.2.3 Proof of Proposition 2.2.2

Proof of Proposition 2.2.2.

We wish to apply the *Implicit Function Theorem* to the f-tension;

$$\tau_\cdot(\cdot) : C^{1,\alpha}\big((\mathcal{M},g);\mathbb{R}\big) \times C^{2,\alpha}\big((\mathcal{M},g);\mathbb{R}^N\big) \to C^{0,\alpha}\big((\mathcal{M},g);\mathbb{R}^N\big)$$

where

$$\tau_f(u) = tr_g\nabla(f du).$$

We calculate the derivative $d_2\left(\tau_f(u)\right)(v)$. Let

$$v,\ w \in C^{2,\alpha}\big((\mathcal{M},g);\mathbb{R}^N\big)$$

be some vector fields (where $\mathbb{R}^N$ is given the standard metric). Let u be f-harmonic and let $u_{s,t} = u + sv + tw$. Therefore

$$\begin{aligned}
\int_{\mathcal{M}} \langle d_2\tau_f(u)(v), w\rangle \ d\mathcal{M} &= \int_{\mathcal{M}} \left\langle \frac{\partial}{\partial s}\tau_f(u_s), \frac{\partial u_{s,t}}{\partial t}\Big|_{t=0}\right\rangle \ d\mathcal{M}\Bigg|_{s=0} \\
&= \frac{d}{ds}\left(\int_{\mathcal{M}} \left\langle \tau_f(u_s), \frac{\partial u_{s,t}}{\partial t}\Big|_{t=0}\right\rangle \ d\mathcal{M}\right)\Bigg|_{s=0} \\
&\quad - \int_{\mathcal{M}} \left\langle \tau_f(u), \frac{\partial}{\partial s}\frac{\partial}{\partial t}u_{s,t}\Big|_{s=t=0}\right\rangle \ d\mathcal{M} \\
&= -\frac{d}{ds}\left(\frac{\partial}{\partial t}E_f(u_{s,t})\Big|_{t=0}\right)\Bigg|_{s=0} - 0
\end{aligned}$$

by definition of τ_f. But

$$\frac{\partial^2}{\partial s \partial t}E_f(u_{s,t})\Bigg|_{s=t=0} = H_u\left(v,w\right) = \int_{\mathcal{M}} \langle J_{f,u}(v), w\rangle \ d\mathcal{M}.$$

Thus

$$d_2\tau_f(u) : C^{2,\alpha}\big((\mathcal{M},g);\mathbb{R}^N)\big) \to C^{0,\alpha}\big((\mathcal{M},g);\mathbb{R}^N\big)$$

is given by

$$d_2\tau_f(u)\big(v\big) = -J_{f,u}v = tr_g\nabla f d^{\nabla}v + fR^{(\mathbb{R}^N,h_1)}(v, du(\partial_\alpha))du(\partial_\alpha) \qquad (2.2.4)$$

if u is f-harmonic.

For general $(f,u) \in C^{1,\alpha}\big(\mathcal{M};(0,\infty)\big) \times C^{2,\alpha}\big((\mathcal{M},g);\mathbb{R}^N\big)$, we use the following formula for the f-tension in coordinates

$$\tau_f(u)^l = f\Delta_{\mathcal{M}}u^l + fg^{\alpha\beta}\Gamma^l_{ij}(u)u^i_\alpha u^j_\beta + g^{\alpha\beta}f_\alpha u^l_\beta.$$

Define now a linear function $\Xi : C^{2,\alpha}\big((\mathcal{M},g);\mathbb{R}^N)\big) \to C^{0,\alpha}\big((\mathcal{M},g);\mathbb{R}^N\big)$ by

$$\Xi^l(v) := f\Delta v^l + fg^{\alpha\beta}\Gamma^l_{ij}(u)\left(u^i_\alpha v^j_\beta + v^i_\alpha u^j_\beta\right) + fg^{\alpha\beta}\left(\Gamma^l_{ij}(u)\right)_q v^q u^i_\alpha u^j_\beta + g^{\alpha\beta}f_\alpha v^l_\beta.$$

It follows that

$$\frac{\tau_f(u+v)^l - \tau_f(u)^l - \Xi(v)}{\|v\|} \to 0$$

as $v \to 0$, so $\tau_\cdot(\cdot)$ is differentiable at (f,u) and $d_2\tau_f(u)(v)^l = \Xi^l(v)$. We must check that $d_2\tau.(\cdot)$ is continuous at each point

$$(f_1, u_1) \in C^{1,\alpha}\big(\mathcal{M}; (0,\infty)\big) \times C^{2,\alpha}\big((\mathcal{M}, g); \mathbb{R}^N\big).$$

Let $\varepsilon' > 0$ and let $v \in C^{2,\alpha}\big((\mathcal{M}, g); \mathbb{R}^N\big)$. Suppose that we have some $(f_2, u_2) \in C^{1,\alpha}\big(\mathcal{M}; (0,\infty)\big) \times C^{2,\alpha}\big((\mathcal{M}, g); \mathbb{R}^N\big)$ satisfying $\|f_1 - f_2\|_{C^{1,\alpha}} < \delta'$ and $\|u_1 - u_2\|_{C^{2,\alpha}} < \delta'$ for some $\delta' > 0$ to be chosen later. Then, for some constant $c = c(g, \mathcal{N}, h_1)$, we have that

$$\begin{aligned}
&\left\|d_2\tau_{f_1}(u_1)(v)^l - d_2\tau_{f_2}(u_2)(v)^l\right\|_{C^{0,\alpha}} \\
&\quad = \left\|(f_1 - f_2)\Delta v^l\right\|_{C^{0,\alpha}} \\
&\qquad + \left\|f_1 g^{\alpha\beta}\Gamma^l_{ij}(u_1)\left(u^i_{1,\alpha}v^j_\beta + v^i_\alpha u^j_{1,\beta}\right) - f_2 g^{\alpha\beta}\Gamma^l_{ij}(u_2)\left(u^i_{2,\alpha}v^j_\beta + v^i_\alpha u^j_{2,\beta}\right)\right\|_{C^{0,\alpha}} \\
&\qquad + \left\|f_1\left(\Gamma^l_{ij}(u_1)\right)_q v^q u^i_{1,\alpha}u^j_{1,\beta} - f_2\left(\Gamma^l_{ij}(u_2)\right)_q v^q u^i_{2,\alpha}u^j_{2,\beta}\right\|_{C^{0,\alpha}} \\
&\qquad + \left\|g^{\alpha\beta}(f_{1,\alpha} - f_{2,\alpha})v^l_\beta\right\|_{C^{0,\alpha}} \\
&\quad \le c\Big\{\|f_1 - f_2\|_{C^{1,\alpha}}\|v\|_{C^{2,\alpha}} \\
&\qquad + \|f_1 - f_2\|_{C^{1,\alpha}}\left\|g^{\alpha\beta}\Gamma^l_{ij}(u_1)\left(u^i_{1,\alpha}v^j_\beta + v^i_\alpha u^j_{1,\beta}\right)\right\|_{C^{0,\alpha}} \\
&\qquad + \|f_2\|_{C^{1,\alpha}}\left\|g^{\alpha\beta}\left(\Gamma^l_{ij}(u_1) - \Gamma^l_{ij}(u_2)\right)\left(u^i_{1,\alpha}v^j_\beta + v^i_\alpha u^j_{1,\beta}\right)\right\|_{C^{0,\alpha}} \\
&\qquad + \|f_2\|_{C^{1,\alpha}}\left\|g^{\alpha\beta}\Gamma^l_{ij}(u_2)\left(u^i_{1,\alpha}v^j_\beta + v^i_\alpha u^j_{1,\beta} - u^i_{2,\alpha}v^j_\beta - v^i_\alpha u^j_{2,\beta}\right)\right\|_{C^{0,\alpha}} \\
&\qquad + \|f_1 - f_2\|_{C^{1,\alpha}}\left\|g^{\alpha\beta}\left(\Gamma^l_{ij}(u_1)\right)_q v^q u^i_{1,\alpha}u^j_{1,\beta}\right\|_{C^{0,\alpha}} \\
&\qquad + \|f_2\|_{C^{1,\alpha}}\left\|g^{\alpha\beta}\left(\left(\Gamma^l_{ij}(u_1)\right)_q v^q - \left(\Gamma^l_{ij}(u_2)\right)_q v^q\right)u^i_{1,\alpha}u^j_{1,\beta}\right\|_{C^{0,\alpha}} \\
&\qquad + \|f_2\|_{C^{1,\alpha}}\left\|g^{\alpha\beta}\left(\Gamma^l_{ij}(u_2)\right)_q v^q\left(u^i_{1,\alpha}u^j_{1,\beta} - u^i_{2,\alpha}u^j_{2,\beta}\right)\right\|_{C^{0,\alpha}} \\
&\qquad + \left\|g^{\alpha\beta}(f_{1,\alpha} - f_{2,\alpha})v^l_\beta\right\|_{C^{0,\alpha}}\Big\} \\
&\quad \le c\Big\{\delta'\|v\|_{C^{2,\alpha}} + \delta'\|u_1\|_{C^{2,\alpha}}\|v\|_{C^{2,\alpha}} \\
&\qquad + (\|f_1\|_{C^{1,\alpha}} + \delta')\left\|\Gamma^l_{ij}(u_1) - \Gamma^l_{ij}(u_2)\right\|_{C^{0,\alpha}}\|u_1\|_{C^{2,\alpha}}\|v\|_{C^{2,\alpha}} \\
&\qquad + (\|f_1\|_{C^{1,\alpha}} + \delta')\delta'\|v\|_{C^{2,\alpha}} + \delta'\|v\|_{C^{2,\alpha}}\|u_1\|^2_{C^{2,\alpha}} \\
&\qquad + (\|f_1\|_{C^{1,\alpha}} + \delta')\left\|\Gamma^l_{ij}(u_1) - \Gamma^l_{ij}(u_2)\right\|_{C^{1,\alpha}}\|v\|_{C^{2,\alpha}}\|u_1\|^2_{C^{2,\alpha}} \\
&\qquad + (\|f_1\|_{C^{1,\alpha}} + \delta')\|v\|_{C^{2,\alpha}}\delta'(\|u_1\|_{C^{2,\alpha}} + \delta') + \delta'\|v\|_{C^{2,\alpha}}\Big\} < \varepsilon'
\end{aligned}$$

for sufficiently small $\delta' > 0$, due to the smoothness of Γ.

It just remains to show that $d_2\tau_f(u)$ is invertible as a map $C^{2,\alpha} \to C^{0,\alpha}$ – then we may use the Implicit Function Theorem.

Suppose that $\phi \in C^{0,\alpha}$. Then $\phi \in L^2$. By Lemma 2.2.5, there are no Jacobi Fields on $(\mathbb{R}^N, h_1)$ – that is no, non-identically zero, solutions to $J_{f,u}^{(\mathbb{R}^N,h_1)}(v) = 0$. Therefore, by the Fredholm Alternative (e.g. (an extension to manifolds of) [Eva99, §6.2 Theorem 4]), there exists a unique weak solution $v \in W^{1,2}(\mathcal{M}; \mathbb{R}^N)$ to

$$J_{f,u}(v) = \phi.$$

It follows by regularity theory that $v \in C^{2,\alpha}(\mathcal{M}; \mathbb{R}^N)$ and so $d_2\tau_f(u)$ is invertible.

We may now apply the Implicit Function Theorem to see that there exist open balls $R := B_\delta^X(f) \subset C^{1,\alpha}\big(\mathcal{M}; (0,\infty)\big) \subset C^{1,\alpha}\big(\mathcal{M}; \mathbb{R}\big)$ and $S := B_\varepsilon^Y(u) \subset C^{2,\alpha}((\mathcal{M}, g); (\mathbb{R}^N))$, and a continuous function $F : R \to S$ such that

$$\tau_{f_1}\big(F(f_1)\big) = 0$$

for all $f_1 \in R$. Thus, for every $f_1 \in R$, there is a unique f_1-harmonic map $u_1 : (\mathcal{M}, g) \to (\mathbb{R}^N, h_1)$ in S.

In the case when $f_1 \in C^\infty(\mathcal{M}, (0,\infty))$, we see by standard regularity theory that $u_1 \in C^\infty((\mathcal{M}, g); (\mathbb{R}^N, h_1))$. Finally, we may assume that $\varepsilon < \rho$, so as to simplify our handling of h_1 and so that we may use Lemma 2.2.4. For sufficiently small ε we see that, like u, u_1 must be non-constant. We can then apply the aforementioned lemma to u_1, to see that u_1 maps into $\mathcal{N}$. $\square$

Chapter 3

Heat Flow

Recall the f-harmonic heat flow equation from chapter 1,

$$\begin{cases} u_t - f(x)\Delta_{\mathcal{M}} u = f(x)A(u)(\nabla u, \nabla u) + \nabla f * \nabla u \\ u|_{t=0} = u_0 \\ u(\cdot, t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}. \end{cases} \tag{1.5.1}$$

In this section we adapt the work of Struwe [Str96] and K.C. Chang [Cha89], on harmonic map heat flow, to the f-harmonic heat flow (1.5.1). The whole of this chapter is devoted to the proof of Theorem 3.1.1.

3.1 The f-Harmonic Heat Flow Theorem

Let $\mathcal{M}$ be a smooth, compact, Riemannian surface.

Theorem 3.1.1 (f-harmonic Heat Flow). *Let $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$. If $\partial\mathcal{M}$ is non-empty, suppose further that $u_0|_{\partial\mathcal{M}} \in C^{2,\alpha}(\partial\mathcal{M};\mathcal{N})$. There exists a weak solution $u : \mathcal{M} \times [0,\infty) \to \mathcal{N}$ of (1.5.1) with the following properties*

(i) u is smooth on $\mathcal{M} \times (0,\infty)$ away from finitely many points $(\bar{x}_k, \bar{t}_k)$, $1 \le k \le K$, $0 < \bar{t}_k \le \infty$;

(ii) $E_f\big(u(t)\big) \le E_f\big(u(s)\big)$ for all $0 \le s \le t$; and

(iii) u assumes the initial data continuously in $W^{1,2}(\mathcal{M},\mathcal{N})$.

The solution u is unique in this class.

Furthermore, at a singular (or bubble) point $(\bar{x},\bar{t}) \in \mathcal{M} \times (0,\infty]$, there exist sequences $x_m \to \bar{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ and a non-constant harmonic map $\tilde{u} : \mathbb{R}^2 \to \mathcal{N}$ with finite (harmonic) energy, such that as $m \to \infty$,

$$u_m(x) := u\big(\exp_{x_m}(R_m x), t_m\big) \to \tilde{u} \tag{3.1.1}$$

in $W^{2,2}_{loc}(\mathbb{R}^2;\mathcal{N})$. *Moreover* $\tilde{u}$ *extends to a smooth harmonic map* $\bar{u} : \mathbb{R}^2 \cup \{\infty\} = S^2 \to \mathcal{N}$ *which we call a 'bubble'.*

There exists a further sequence of times $t_m \to \infty$ *such that the sequence of maps* $u(\cdot, t_m)$ *converges weakly in* $W^{1,2}(\mathcal{M};\mathcal{N})$ *to a smooth f-harmonic map* $u_\infty : \mathcal{M} \to \mathcal{N}$*, and smoothly away from finitely many points* $\bar{x}_k$*.*

For simplicity, we only give a proof here in the case of domains without boundary. Precisely:

Theorem 3.1.2. *Let* $\mathcal{M}$ *be a smooth, compact, Riemannian surface without boundary. Let* $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$*. There exists a weak solution* $u : \mathcal{M} \times [0,\infty) \to \mathcal{N}$ *of (1.5.1) with the following properties*

(i) u *is smooth on* $\mathcal{M} \times (0,\infty)$ *away from finitely many points* $(\bar{x}_k, \bar{t}_k)$, $1 \le k \le K$, $0 < \bar{t}_k \le \infty$*;*

(ii) $E_f\big(u(t)\big) \le E_f\big(u(s)\big)$ *for all* $0 \le s \le t$*; and*

(iii) u *assumes the initial data continuously in* $W^{1,2}(\mathcal{M},\mathcal{N})$*.*

The solution u *is unique in this class.*

Furthermore, at a singular (or bubble) point $(\bar{x}, \bar{t}) \in \mathcal{M} \times (0,\infty]$*, there exist sequences* $x_m \to \bar{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ *and a non-constant harmonic map* $\tilde{u} : \mathbb{R}^2 \to \mathcal{N}$ *with finite (harmonic) energy, such that as* $m \to \infty$,

$$u_m(x) := u\big(\exp_{x_m}(R_m x), t_m\big) \to \tilde{u} \tag{3.1.2}$$

in $W^{2,2}_{loc}(\mathbb{R}^2;\mathcal{N})$. *Moreover* $\tilde{u}$ *extends to a smooth harmonic map* $\bar{u} : \mathbb{R}^2 \cup \{\infty\} = S^2 \to \mathcal{N}$ *which we call a 'bubble'.*

There exists a further sequence of times $t_m \to \infty$ *such that the sequence of maps* $u(\cdot, t_m)$ *converges weakly in* $W^{1,2}(\mathcal{M};\mathcal{N})$ *to a smooth f-harmonic map* $u_\infty : \mathcal{M} \to \mathcal{N}$*, and smoothly away from finitely many points* $\bar{x}_k$*.*

The proof of Theorem 3.1.2 is given in section 3.4 with some technical results given beforehand in section 3.3.

3.2 A remark about Theorem 3.1.1

The reader should note that the f-harmonic Heat Flow Theorem asserts that if a bubble should form, then that bubble is a *harmonic* (1-harmonic) bubble. So despite extending the theory from harmonic map heat flow to f-harmonic map heat flow, we find the type of singularities involved remain of the same type. This is to be expected.

Indeed, suppose that we consider a singularity forming at the point $(\bar{z}, \bar{t})$. To "examine" this bubble, we blow-up near $\bar{z}$. But as we do this, we are in effect "flattening the function f to a constant function". More precisely, consider $u_m : \mathbb{R}^2 \to \mathcal{N}$ given by $u_m(x) := u\big(\exp_{z_m}(R_m x), t_m\big)$ (where $z_m \to \bar{z}$, $t_m \nearrow \bar{t}$ and $R_m \searrow 0$ as $m \to \infty$), and then consider the functions

$$f_m(x) := f\big(\exp_{z_m}(R_m x)\big). \tag{3.2.1}$$

Notice that $\nabla f_m \to 0$ as $m \to \infty$. Intuitively as $m \to \infty$, we get "closer and closer" to the (1-)harmonic case. Therefore it is not surprising that we find *harmonic* bubbles.

3.3 Regularity

Notice first that if u is a solution of (1.5.1) then

$$|\partial_t u - f\Delta u| \leq C\left(|\nabla u| + |\nabla u|^2\right) \leq C\left(1 + |\nabla u|^2\right). \tag{3.3.1}$$

We shall use this fact later.

We quote now the following lemma (see [Str96, Lemma III.6.7]) which we shall need.

Lemma 3.3.1. *Suppose $\phi \in C_c^\infty(B_R)$ satisfies $0 \leq \phi \leq 1$ and $|\nabla\phi| \leq \frac{4}{R}$. Then for any function $v \in W^{1,2}_{loc}(\mathbb{R}^2)$, we have*

$$\int_{\mathbb{R}^2} |v|^4\phi^2 \leq c_0 \left(\int_{B_R} |v|^2 dx\right)\left(\int_{B_R} |\nabla v|^2\phi^2 dx + R^{-2}\int_{B_R} |v|^2 dx\right), \tag{3.3.2}$$

where c_0 is independent of both v and R.

The four results in this section are analogous to four results of Struwe for harmonic maps – namely [Str96, Lemmata III.6.8-11] respectively – and the proofs are fundamentally the same, but with perhaps extra terms to keep track of.

Lemma 3.3.2. *Suppose $u \in C^2(\mathcal{M}\times[0,T);\mathcal{N})$ is a solution of (1.5.1). Then*

$$E_f(u(t)) + \int_0^t \int_{\mathcal{M}} |\partial_t u|^2 d\mathcal{M} dt = E_f(u_0) \tag{3.3.3}$$

for any $t < T$.

Proof.
By (1.5.1), at time t

$$\begin{aligned}\frac{d}{dt}E_f(u(t)) &= \int_{\mathcal{M}} f(x)\,\langle \nabla\partial_t u, \nabla u\rangle\, d\mathcal{M}\\ &= -\int_{\mathcal{M}} \big\langle \partial_t u, \operatorname{div}\big(f\nabla u\big)\big\rangle\, d\mathcal{M}\\ &= -\int_{\mathcal{M}} |\partial_t u|^2 d\mathcal{M}.\end{aligned}$$

The assertion follows by integrating over $[0,t]$. $\square$

Let $\imath_{\mathcal{M}}$ denote the injectivity radius of the exponential map on $\mathcal{M}$ – that is, let $\imath_{\mathcal{M}}$ be the largest number such that $\exp : B^{\mathbb{R}^2}_{\imath_{\mathcal{M}}}(0) \to \mathcal{M}$ is a diffeomorphism onto its image. By using a conformal change of variables, we may use Euclidean coordinates on any ball $B_R(x_0) \subset \mathcal{M}$. Define

$$E_f(u; B_R(x_0)) := \frac{1}{2}\int_{B_R(x_0)} f(x)|\nabla u|^2 dx, \qquad R < \imath_{\mathcal{M}}. \tag{3.3.4}$$

Set $B_R := B_R(0)$. The following lemma is analogous to [Str96, Lemma III.6.9].

Lemma 3.3.3. *There exists $c_1 = c_1(f)$ such that, for all $R < \frac{1}{2}\imath_{\mathcal{M}}$, if $u \in C^2(B_{2R}\times[0,T];\mathcal{N})$ is a solution to (1.5.1) with $E_f(u(t); B_{2R}) \le E_0$ for some constant E_0, then*

$$E_f(u(T); B_R) \le E_f(u_0; B_{2R}) + \frac{c_1 T E_0}{R^2}. \tag{3.3.5}$$

Proof.
We start by choosing a function $\phi \in C_c^\infty(B_{2R};[0,1])$ such that $\phi|_{B_R} \equiv 1$ and $|\nabla\phi| \le \frac{2}{R}$. Multiplying the first line of (1.5.1) by $\phi^2\partial_t u$ and integrating

by parts, we see

$$\begin{aligned}\int_0^T \int_{B_{2R}} &\left\{ |\partial_t u|^2 \phi^2 + \frac{1}{2}\frac{d}{dt}\big(f(x)|\nabla u|^2\phi^2\big)\right\} dxdt \\ &= -2\int_0^T \int_{B_{2R}} f(x)\phi\partial_t u \nabla\phi\nabla u \; dxdt \\ &\le \int_0^T \int_{B_{2R}} |\partial_t u|^2\phi^2 dxdt + \int_0^T \int_{B_{2R}} f^2|\nabla u|^2|\nabla\phi|^2 dxdt\end{aligned}$$

by Cauchy's Inequality. So

$$\begin{aligned}E_f(u(T); B_R) - E_f(u_0; B_{2R}) &\le \frac{1}{2}\int_{B_{2R}} f|\nabla u|^2\phi^2 dxdt\Big|_{t=0}^T \\ &\le \frac{4}{R^2}\int_0^T \int_{B_{2R}} f^2|\nabla u|^2 dxdt \\ &\le \frac{4}{R^2} T E_0 \sup_{x\in\mathcal{M}} f(x)\end{aligned}$$

as claimed. □

The following lemma is analogous to [Str96, Lemma III.6.10].

Lemma 3.3.4. *There exists $\varepsilon_1 = \varepsilon_1(\mathcal{M},\mathcal{N}) > 0$ such that if, for some $R \in (0, \frac{1}{2}\imath_{\mathcal{M}})$, $u \in C^2(B_{2R}\times[0,T),\mathcal{N})$ satisfies,*

(i) $E_f(u(t)) \le E_0$;

(ii) *u solves $u_t - f\Delta_{\mathcal{M}} u = fA(u)(\nabla u, \nabla u) + \nabla f * \nabla u$ on $B_{2R}\times[0,T)$;*

(iii) $\sup_{(x,t)\in B_{2R}\times[0,T]} E_f\big(u(t); B_R(x)\big) < \varepsilon_1$,

then

$$\int_0^T \int_{B_R} |\nabla^2 u|^2 dx\; dt \le cE_0\left(1 + T + TR^{-2}\right) \tag{3.3.6}$$

for some $c = c(\mathcal{M},\mathcal{N},f)$.

Proof.

Set $Q := B_{2R}\times[0,T)$. Let $\phi \in C_c^\infty(B_{2R};[0,1])$ satisfy $\phi|_{B_R} = 1$ and $|\nabla\phi| \le \frac{2}{R}$. First, we use the binomial inequality, $2|ab| \le \delta a^2 + \delta^{-1}b^2$, to see that

$$\begin{aligned}\int_Q |\nabla^2 u|^2\phi^2 &= -\int_Q \nabla u \nabla \Delta u\ \phi^2 - 2\int_Q \nabla u \nabla^2 u\ \phi\nabla\phi \\ &= \int_Q \Delta u \Delta u\ \phi^2 + 2\int_Q \nabla u \Delta u\ \phi\nabla\phi - 2\int_Q \nabla u\nabla^2 u\ \phi\nabla\phi \\ &\le \int_Q |\Delta u|^2\phi^2 + \gamma\int_Q |\Delta u|^2\ \phi^2 + \gamma^{-1}\int_Q |\nabla u|^2|\nabla\phi|^2 \\ &\quad + \delta\int_Q |\nabla^2 u|^2\phi^2 + \delta^{-1}\int_Q |\nabla u|^2|\nabla\phi|^2\end{aligned}$$

for any $\gamma, \delta > 0$. It follows that

$$\int_Q |\nabla^2 u|^2\phi^2 dx\ dt \le 2\int_Q |\Delta u|^2\phi^2 dx\ dt + cR^{-2}\int_Q |\nabla u|^2 dx\ dt. \tag{3.3.7}$$

Now, since $\partial_t u\ \nabla u = (f\Delta u + \nabla f * \nabla u)\nabla u$, we see from (3.3.1) that

$$\begin{aligned}&\int_Q \partial_t\left(\frac{1}{2}f(x)|\nabla u|^2\phi^2\right) + |\Delta u|^2\phi^2\ dxdt = \\ &= \int_Q \left(|\Delta u|^2 - |\partial_t u|^2\right)\phi^2 - 2f\nabla u\ \partial_t u\ \nabla\phi\ \phi\ dxdt \\ &\le c\int_Q \Big|f\Delta u - \partial_t u\Big|\Big|f\Delta u + \partial_t u\Big|\phi^2\ dx\ dt \\ &\quad - 2\int_Q f\nabla u\ \partial_t u\ \nabla\phi\ \phi\ dxdt \\ &\le c\int_Q \Big[|\nabla u| + |\nabla u|^2\Big]\Big[|f\Delta u| + |\nabla f * \nabla u|\Big]\phi^2\ dx\ dt \\ &\quad - 2\int_Q f\nabla u(f\Delta u + \nabla f * \nabla u)\nabla\phi\ \phi\ dx\ dt \\ &\le c\int_Q \Big[|\nabla u| + |\nabla u|^2\Big]^2\phi^2\ dx\ dt + c\int_Q f^2|\nabla u|^2|\nabla\phi|^2\ dx\ dt \\ &\quad + \frac{1}{2}\int_Q |\Delta u|^2\phi^2\ dx\ dt + c\int_Q |\nabla f * \nabla u|^2\ \phi^2\ dx\ dt \\ &\le c\int_Q \Big\{2|\nabla u|^2 + 2|\nabla u|^4\Big\}\phi^2\ dx\ dt + cR^{-2}\int_Q f|\nabla u|^2\ dx\ dt \\ &\quad + \frac{1}{2}\int_Q |\Delta u|^2\phi^2\ dx\ dt + c\int_Q |\nabla u|^2\ \phi^2\ dx\ dt\end{aligned}$$

Here we have used that $f\Delta u+\partial_t u = f\Delta u+f(\Delta u)^\top+\nabla f*\nabla u$, where $(\Delta u)^\top$ is the component of Δu in $T_u\mathcal{N}$. We now apply Lemma 3.3.1 to see that

$$\begin{aligned}
\frac{1}{2}\int_Q |\Delta u|^2\phi^2\;dx\;dt \le c\int_Q |\nabla u|^4\phi^2\;dx\;dt& \\
+c\left(1+R^{-2}\right)\int_Q f|\nabla u|^2\;dx\;dt& \\
-\int_Q \partial_t\left(\frac{1}{2}f(x)|\nabla u|^2\phi^2\right)\;dx\;dt& \\
\le c_1\varepsilon_1\left(\int_Q |\nabla^2 u|^2\phi^2 dx\;dt + R^{-2}\int_Q|\nabla u|^2 dx\;dt\right)& \\
+c\left(1+R^{-2}\right)\int_Q f|\nabla u|^2\;dx\;dt + E_0& \\
\le c_1\varepsilon_1\int_Q |\nabla^2 u|^2\phi^2 dx\;dt + c\left(1+T+TR^{-2}\right)E_0.&
\end{aligned}$$

Therefore, if we choose $\varepsilon_1>0$ such that $c_1\varepsilon_1<\frac{1}{8}$, by (3.3.7) we obtain

$$\int_0^T\int_{B_R}|\nabla^2 u|^2 dx\;dt \le \int_Q |\nabla^2 u|^2\phi^2 dx\;dt \le cE_0\left(1+T+TR^{-2}\right). \quad (3.3.8)$$

□

The next lemma is analogous to [Str96, Lemma III.6.11].

Lemma 3.3.5. *If $u\in C^\infty(B_R\times(0,T);\mathcal{N})$ solves (1.5.1) and satisfies $|\partial_t u|$, $|\nabla^2 u|\in L^2(B_R\times[0,T))$ then u extends smoothly to $B_R\times(0,T]$.*

Proof.

As is [Str96] we assume here for simplicity, that $\mathcal{M}$ is the torus T^2. This avoids the need to use test functions. So as to avoid confusion between the time T and the torus T^2, we denote the torus by Σ.

First notice that for any $t \in (0,T)$ and $s \geq 2$ there holds

$$\begin{aligned}
&\int_\Sigma \Big(f\Delta u - u_t\Big)\nabla\Big(\nabla u|\nabla u|^{s-2}\Big)\ dx \ = \\
&= \int_\Sigma \nabla u_t \nabla u|\nabla u|^{s-2} - \nabla f \Delta u \nabla u|\nabla u|^{s-2} - f\nabla\Delta u\nabla u|\nabla u|^{s-2}\ dx \\
&= \int_\Sigma \Bigg\{\partial_t\Bigg(\frac{|\nabla u|^s}{s}\Bigg) + (s-1)f|\nabla^2 u|^2|\nabla u|^{s-2} \\
&\qquad\qquad + \Delta f|\nabla u|^s + s\nabla f\nabla u\nabla^2 u|\nabla u|^{s-2}\Bigg\}\ dx.
\end{aligned} \tag{3.3.9}$$

So, by (3.3.1), it follows that

$$\begin{aligned}
&\int_\Sigma \Bigg\{\partial_t\Bigg(\frac{|\nabla u|^s}{s}\Bigg) + (s-1)f|\nabla^2 u|^2|\nabla u|^{s-2}\ dx \\
&\leq c\int_\Sigma \Big(|\nabla u|^2 + |\nabla u|\Big)|\nabla^2 u|\ |\nabla u|^{s-2}\ dx - \int_\Sigma \Delta f|\nabla u|^s\ dx \\
&\quad - s\int_\Sigma \nabla f\nabla u\nabla^2 u|\nabla u|^{s-2}\ dx \\
&\leq \gamma\int_\Sigma |\nabla^2 u|^2|\nabla u|^{s-2}\ dx + c\int_\Sigma |\nabla u|^{s-2}\Big(|\nabla u|^2 + |\nabla u|\Big)^2\ dx \\
&\quad + c\int_\Sigma |\nabla u|^s\ dx + c\int_\Sigma |\nabla^2 u||\nabla u|^{s-1}\ dx \\
&\leq \gamma'\int_\Sigma |\nabla^2 u|^2|\nabla u|^{s-2}\ dx + c\int_\Sigma |\nabla u|^{s+2}\ dx + c\int_\Sigma |\nabla u|^s\ dx,
\end{aligned} \tag{3.3.10}$$

where we have complete control over the value of γ'; and where we have used both $2ab \leq \delta a^2 + \delta^{-1}b^2$ and $(a+b)^2 \leq 2a^2 + 2b^2$.

For $0 < t_0 < t_1 < T$, with careful choice of $\gamma' = \gamma'(f)$, and after integrating over t, we may write

$$\begin{aligned}
&\sup_{t_0<t<t_1}\int_\Sigma |\nabla u(t)|^s\, dx + \int_{t_0}^{t_1}\int_\Sigma \left|\nabla\big(|\nabla u|^{s/2}\big)\right|^2 dx\, dt\\
&\le c\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^{s+2}\, dx\, dt + c\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^s\, dx\, dt + \int_\Sigma |\nabla u(t_0)|^s\, dx\\
&\le c\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^{2s}\, dx\, dt\Big)^{\frac12}\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^4\, dx\, dt\Big)^{\frac12}\\
&\quad + c\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^s\, dx\, dt + \int_\Sigma |\nabla u(t_0)|^s\, dx,
\end{aligned} \tag{3.3.11}$$

with constants depending only on $\mathcal{M}$, $\mathcal{N}$, s and f. So if we impose the condition $t_1 - t_0 < \delta$, for sufficiently small δ we may include the $\iint |\nabla u|^s$ term with the $(\sup\int |\nabla u(t)|^s)$ term, and hence write

$$\begin{aligned}
&\sup_{t_0<t<t_1}\int_\Sigma |\nabla u(t)|^s\, dx + \int_{t_0}^{t_1}\int_\Sigma \left|\nabla\big(|\nabla u|^{s/2}\big)\right|^2 dx\, dt\\
&\le c\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^{2s}\, dx\, dt\Big)^{\frac12}\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^4\, dx\, dt\Big)^{\frac12}\\
&\quad + c\int_\Sigma |\nabla u(t_0)|^s\, dx.
\end{aligned} \tag{3.3.12}$$

This is almost exactly the equation arrived at by Struwe (see [Str96, page 228] – the only difference being the constant c appearing in the final term). The f causes us no more problems in this proof, and we proceed almost exactly as in [Str96].

Now, by Lemma 3.3.1

$$\begin{aligned}
\|v\|_{L^{2s}(Q)}^{2s} &= \left\|v^{s/2}\right\|_{L^4(Q)}^4\\
&\le c_0 \sup_{t_0\le t\le t_1}\left\|\big(v(t)\big)^{s/2}\right\|_{L^2(\Sigma)}^2\left(\left\|\nabla\big(v^{s/2}\big)\right\|_{L^2(Q)}^2 + \left\|v^{s/2}\right\|_{L^2(Q)}^2\right)
\end{aligned}$$

where $Q = \Sigma\times[t_0, t_1]$, for any $s \ge 2$. Letting $v = |\nabla u|$ and assuming that

$t_1 - t_0 \le \min\{1, \delta\}$, we obtain from (3.3.12)

$$\begin{aligned}
&\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^{2s}\,dx\,dt \\
&\qquad \le c\Big(\sup_{t_0<t<t_1}\int_\Sigma \big|\nabla u(t)\big|^s\,dx\,dt\Big)\cdot \\
&\qquad\quad \cdot\Big(\sup_{t_0<t<t_1}\int_\Sigma \big|\nabla u(t)\big|^s\,dx\,dt + \int_{t_0}^{t_1}\int_\Sigma \big|\nabla(|\nabla u|^{s/2})\big|^2\,dx\,dt\Big) \\
&\qquad \le c\Big[\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^{2s}\,dx\,dt\Big)^{\frac12}\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^4\,dx\,dt\Big)^{\frac12} \\
&\qquad\quad + \int_\Sigma |\nabla u(t_0)|^s\,dx\Big]^2 \\
&\qquad \le c\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^{2s}\,dx\,dt\Big)\Big(\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^4\,dx\,dt\Big) \\
&\qquad\quad + c\Big(\int_\Sigma |\nabla u(t_0)|^s\,dx\Big)^2
\end{aligned}$$

for any $s \ge 2$. Using Lemma 3.3.1 again, we see that for any $\varepsilon > 0$, we may choose $0 < \delta' = \delta'(f, s, \mathcal{M}, \mathcal{N}) \le \min\{1, \delta\}$ such that if $0 < t_0 < t_1 < T$ and $t_1 - t_0 < \delta'$ then

$$\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^4\,dx\,dt \le c_0 E(u_0)\int_{t_0}^{t_1}\int_\Sigma \big(|\nabla^2 u|^2 + |\nabla u|^2\big)\,dx\,dt < \varepsilon. \quad (3.3.13)$$

Therefore for sufficiently small δ', we have the estimate

$$\int_{t_0}^{t_1}\int_\Sigma |\nabla u|^{2s}\,dx\,dt \le c\Big(\int_\Sigma |\nabla u(t_0)|^s\,dx\Big)^2 \quad (3.3.14)$$

for any $s \ge 2$ and $0 < t_0 < t_1 < t_0 + \delta'$, where $c = c(s, f, \mathcal{M}, \mathcal{N})$. This means that $\nabla u \in L^q(\Sigma \times [t_0, T])$ for any $q < \infty$ and any $t_0 > 0$. But then linear theory tells us that $|\partial_t u|,\ |\nabla^2 u| \in L^q(\Sigma \times [t_0, T])$ for any $q < \infty$. Hence, by a result of Ladyženskaya-Solonnikov-Ural'ceva [LSU68, Corollary of Theorem IV.9.1], ∇u is Hölder continuous on $\Sigma \times [t_0, T]$, for any $t_0 > 0$.

We may then use a "boot-strapping" argument to obtain higher regularity – see for example [Kry96, §8.12] for more details.

□

3.4 Proof of Theorem 3.1.2

The following five propositions provide a proof of the theorem. In this section, the notation c_1 and ε_1 refer to the constants from Lemmata 3.3.3 and 3.3.4 respectively. The proofs of Propositions 3.4.1 – 3.4.5 are almost identical to parts (°1) – (°5) of the proof of [Str96, Theorem III.6.6].

3.4.1 Smooth initial condition

Proposition 3.4.1. *Suppose $u_0 \in C^\infty(\mathcal{M},\mathcal{N})$. Then, for some $T > 0$, there exists a solution*

$$u \in C^\infty(\mathcal{M} \times [0,T] \setminus \{(x_1,T),\dots,(x_K,T)\};\mathcal{N})$$

of (1.5.1), for finite $K \geq 0$ and points $x_1,\dots,x_K \in \mathcal{M}$.

Proof.

By standard results (see for example [Ham75, page 122]), we know that (1.5.1) has a local solution $u \in C^\infty(\mathcal{M} \times [0,T);\mathcal{N})$ for some $T > 0$. By Lemma 3.3.2, it follows that $\partial_t u \in L^2(\mathcal{M}\times[0,T])$ and $E_f(u(t)) \leq E_f(u(s)) \leq E_f(u_0)$ whenever $0 \leq s \leq t \leq T$.

Furthermore by Lemma 3.3.4, *if* it is possible to choose a number $R > 0$ such that

$$\sup_{x\in\mathcal{M},\ t\in[0,T]} E_f(u(t);B_R(x)) < \varepsilon_1$$

then it follows that

$$\int_0^T \int_{B_R} |\nabla^2 u|^2 dx\ dt \leq cE_0\left(1 + T + TR^{-2}\right).$$

We would then see from Lemma 3.3.5 that u extends to a C^∞-solution of (1.5.1) on $\mathcal{M} \times [0,T]$. Therefore, if we knew that u could not be extended smoothly in this way, then the set

$$\left\{x \in \mathcal{M} : \forall\ R > 0,\ \limsup_{t\nearrow T} E_f\big(u(t);B_R(x)\big) \geq \varepsilon_1\right\}$$

must be nonempty. We pick any finite subset $\{x_k\}_{k=1}^K$. Fix $R > 0$. Choose $t_k < T$ for each $k = 1,\dots,K$, such that

$$E_f\big(u(t_k);B_R(x_k)\big) \geq \frac{\varepsilon_1}{2}.$$

Without loss of generality, we may assume that $B_{2R}(x_i) \cap B_{2R}(x_j) = \emptyset$

whenever $i \neq j$, and $t_k \geq T - \frac{\varepsilon_1 R^2}{4c_1 E_0} =: \sigma$. By Lemma 3.3.3 we see that

$$\begin{aligned} E_f\big(u(\sigma)\big) &\geq \sum_{k=1}^{K} E_f\big(u(\sigma); B_{2R}(x_k)\big) \\ &\geq \sum_{k=1}^{K} \left(E_f\big(u(t_k); B_R(x_k)\big) - c_1 \frac{t_k - \sigma}{R^2} E_0 \right) \\ &\geq K\left(\frac{\varepsilon_1}{2} - c_1 \frac{t_k - \sigma}{R^2} E_0 \right) \geq \frac{K\varepsilon_1}{4}. \end{aligned} \tag{3.4.1}$$

But Lemma 3.3.2 gives $E_f\big(u(\sigma)\big) \leq E_f(u_0)$, so we have an upper bound

$$K \leq \frac{4E_f(u_0)}{\varepsilon_1}$$

for the number of singular points $x_1, \ldots, x_K$ at $t = T$.

Now suppose $Q \subset\subset \mathcal{M} \times [0,T] \setminus \{(x_1,T), \ldots, (x_K,T)\}$. There exists $R = R(Q) > 0$ such that

$$\sup_{(x,t)\in Q} E_f\big(u(t); B_R(x)\big) < \varepsilon_1.$$

Then as previously discussed; by Lemmata 3.3.4 and 3.3.5, the solution u extends to a C^∞-solution of (1.5.1) on $\mathcal{M} \times [0,T] \setminus \{(x_1,T), \ldots, (x_K,T)\}$.

□

3.4.2 General initial condition

Proposition 3.4.2. *Suppose $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$. There exists a weak solution u to (1.5.1) which is smooth on $\mathcal{M} \times (0,\infty)$ away from finitely many points $(\bar{x}_k, \bar{t}_k)$, $1 \leq k \leq K$, $0 < \bar{t}_k \leq \infty$. Furthermore u attains its initial data u_0 continuously in $W^{1,2}(\mathcal{M};\mathcal{N})$.*

Proof.

Choose a sequence $u_{0m} \in C^\infty(\mathcal{M};\mathcal{N})$ approximating u_0 in $W^{1,2}$ – this is possible by [Str96, Theorem III.6.2]. For each m, let u_m denote a solution to (1.5.1) with initial condition u_{0m}. Let $T_m > 0$ denote the maximum time of u_m. Such solutions exist by Proposition 3.4.1. Choose $R_0 > 0$ such that

$$\sup_{x\in\mathcal{M}} E_f\big(u_0; B_{2R_0}(x)\big) \leq \frac{\varepsilon_1}{4}.$$

Then this inequality will also hold with u_{0m} in place of u_0, for $m \geq m_0$ (for some m_0), but with $\frac{\varepsilon_1}{4}$ replaced by of $\frac{\varepsilon_1}{2}$. Hence, for $T = \frac{\varepsilon_1 R_0^2}{4c_1 E(u_0)}$, we have

$$\begin{aligned}\sup_{x\in\mathcal{M},\ 0\le t\le\min\{T_m,T\}} E_f\big(u_m(t);B_{R_0}(x)\big) &\le \sup_{x\in\mathcal{M}} E_f\big(u_0;B_{2R_0}(x)\big) + c_1\frac{T}{R_0^2}E(u_0)\\ &\le \frac{\varepsilon_1}{2}+\frac{\varepsilon_1}{4} < \varepsilon_1\end{aligned}$$

by Lemma 3.3.3. It then follows by Lemma 3.3.4 that $\nabla^2 u_m$ is uniformly bounded in $L^2(\mathcal{M}\times[0,t])$ for $t\le\min\{T,T_m\}$, in terms of $E_f(u_0)$ and R_0 only. Therefore, by Lemma 3.3.5, the interval of existence (of u_m) is both open and closed in $[0,T]$; i.e. $T_m \ge T > 0$.

Furthermore

$$\int_0^T\int_{B_R} |\nabla^2 u_m|^2 dx\ dt \le cE_f(u_0)\left(1+T+TR_0^{-2}\right)$$

uniformly. Using Lemma 3.3.2 in addition, we may assume that u_m converges weakly to a solution u of (1.5.1) such that $E_f(u(t))\le E_f(u_0)$ for $t\in[0,T]$, and $|\partial_t u|$, $|\nabla^2 u|\in L^2(\mathcal{M}\times[0,T])$. Since $|\partial_t u|\in L^2(\mathcal{M}\times[0,T])$, u attains the initial condition u_0 continuously in $L^2(\mathcal{M};\mathcal{N})$. Moreover, by the uniform energy bound $E_f(u(t))\le E_f(u_0)$, this also holds in $W^{1,2}(\mathcal{M};\mathcal{N})$.

By Lemma 3.3.5 we see that $u\in C^\infty(\mathcal{M}\times(0,T_1),\mathcal{N})$ for some maximal $T_1>T$. Then by Lemma 3.3.2, we have $E_f\big(u(t)\big)\le E_f\big(u(s)\big)\le E_f\big(u_0\big)$ for all $0\le s\le t<T_1$. By Proposition 3.4.1, u extends smoothly to $\mathcal{M}\times(0,T_1]\setminus\big\{(x_1,T_1),\dots,(x_{K_1},T_1)\big\}$ for some finite collection of singular points x_k, $k=1,\dots,K_1$. Furthermore, as $t\nearrow T_1$ we have that $u(t)\to u_0^{(1)}\in W^{1,2}(\mathcal{M};\mathcal{N})$ weakly, and strongly in $W^{1,2}_{loc}(\mathcal{M}\setminus\{x_1,\dots,x_{K_1}\};\mathcal{N})$. Hence by (3.4.1),

$$\begin{aligned}E_f\big(u_0^{(1)}\big) &= \lim_{R\to0} E_f\left(u_0^{(1)};\mathcal{M}\setminus\bigcup_{k=1}^{K_1} B_{2R}(x_k)\right)\\ &\le \lim_{R\to0}\limsup_{t\nearrow T_1}\left(E_f\big(u(t);\mathcal{M}\big)-\sum_{k=1}^{K_1}E_f\big(u(t);B_{2R}(x_k)\big)\right)\\ &\le E_f(u_0)-\frac{K_1\varepsilon_1}{4}.\end{aligned}$$

Let $u^{(0)}=u$ and $T_0=0$. Using iteration, we get a sequence $u^{(m)}$ of solutions to (1.5.1) on $\mathcal{M}\times(T_m,T_{m+1})$ with initial data $u_0^{(m)}$ satisfying $u^{(m)}(t)\to u_0^{(m+1)}$ weakly in $W^{1,2}(\mathcal{M};\mathcal{N})$ as $t\nearrow T_{m+1}$. For each $m\in\mathbb{N}$, there are finitely many singularities $x_1^{(m)}, \ldots, s_{K_{m+1}}^{(m)}$ of $u^{(m)}$ at $t = T_{m+1}$. Note that

$$\sum_{l=1}^{m+1} K_l \varepsilon_1/4 \leq E_f(u_0).$$

Moreover $u^{(m)}(t) \to u_0^{(m+1)}$ smoothly away from $\{x_1^{(m)}, \ldots, s_{K_{m+1}}^{(m)}\}$ as $t \nearrow T_{m+1}$. In particular, the total number of singularities of the sequence $u^{(m)}$ is finite. If we piece the $u^{(m)}$ together, we obtain a weak solution u to (1.5.1) which (for any initial $u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$) is smooth on $\mathcal{M} \times (0, \infty)$ away from finitely many points. $\square$

3.4.3 Asymptotics

Proposition 3.4.3. *Suppose $u_0 \in W^{1,2}(\mathcal{M}; \mathcal{N})$ and let u be a weak solution to (1.5.1) which is smooth on $\mathcal{M} \times (0, \infty)$ away from finitely many points. Then there exists a sequence $t_m \to \infty$ such that $u(\cdot, t_m)$ converges weakly in $W^{1,2}(\mathcal{M}, \mathcal{N})$ to a smooth f-harmonic map $u_\infty : \mathcal{M} \to \mathcal{N}$; and smoothly away from finitely many points.*

Proof.

1. Suppose first that for some $T > 0$ and $R > 0$ we have

$$\sup_{x \in \mathcal{M}, t > T} E_f\big(u(t); B_R(x)\big) < \varepsilon_1.$$

Then, by Lemma 3.3.4, for any $t > T$ it follows that

$$\int_t^{t+1} \int_{\mathcal{M}} f(x) |\nabla^2 u|^2 dx \; dt \leq c E_f(u_0)(1 + R^{-2})$$

with $c = c(f, \mathcal{M}, \mathcal{N})$. Moreover, by Lemma 3.3.2

$$\int_t^{t+1} \int_{\mathcal{M}} |\partial_t u|^2 dx \; dt \to 0$$

as $t \to \infty$. Thus there exists a sequence $t_m \to \infty$ such that $u_m := u(t_m) \rightharpoonup u_\infty$ weakly in $W^{2,2}(\mathcal{M}; \mathcal{N})$, and $\partial_t u(t_m) \to 0$ in L^2. Moreover, by Rellich–Kondrakov, $u_m = u(t_m) \to u_\infty$ strongly in $W^{1,p}(\mathcal{M}; \mathcal{N})$ for any $p < \infty$.

Now, by (1.5.1)

$$\begin{aligned} \Delta\left(u_m - u_\infty\right) = &\frac{1}{f} \partial_t u(t_m) - \frac{1}{f} \nabla f * -\nabla u_m \\ &- A(u_m)(\nabla u_m, \nabla u_m) - \Delta u_\infty. \end{aligned}$$

So

$$\begin{aligned}
&\int_{\mathcal{M}} \left|\nabla^2(u_m - u_\infty)\right|^2 \, dx \\
&\quad \le \left| \int_{\mathcal{M}} \frac{1}{f}\partial_t u(t_m)\Delta\left(u_m - u_\infty\right) \, dx \right| + \left| \int_{\mathcal{M}} \frac{1}{f}\nabla f * \nabla u_m \, \Delta\left(u_m - u_\infty\right) \, dx \right| \\
&\qquad + \left| \int_{\mathcal{M}} A(u_m)(\nabla u_m, \nabla u_m)\Delta\left(u_m - u_\infty\right) \, dx \right| + \left| \int_{\mathcal{M}} |\Delta u_\infty \, \Delta\left(u_m - u_\infty\right) \, dx \right| \\
&\quad \to 0.
\end{aligned}$$

Hence we also have $u(t_m) \to u_\infty$ strongly in $W^{2,2}(\mathcal{M};\mathcal{N})$. Taking limits in (1.5.1), we see that u_∞ is f-harmonic.

2. We now study the remaining case. Suppose that $t_m \to \infty$ such that $u(t_m) \rightharpoonup \tilde{u}_\infty$ weakly in $W^{1,2}(\mathcal{M},\mathcal{N})$ and suppose that there exist (a finite number of) points $x_1, \dots, x_K$ such that

$$\liminf_{m\to\infty} \left(E_f(u(t_m); B_R(x_k))\right) \ge \frac{\varepsilon_1}{2}, \tag{3.4.2}$$

for all $R > 0$ and $1 \le k \le K$. Choosing $R > 0$ such that $B_{2R}(x_j) \cap B_{2R}(x_k) = \emptyset$ whenever $x_j \ne x_k$, we have

$$E_f\big(u(t_m)\big) \ge \sum_{k=1}^{K} E_f\big(u(t_m); B_R(x_k)\big) \ge \frac{K\varepsilon_1}{4}$$

for sufficiently large m. Hence $K \le 4E_f(u_0)/\varepsilon_1$. So let $x_1, \dots, x_K$ denote *all* the points where (3.4.2) holds. Now, by Lemma 3.3.3, for all $x \notin \{x_1, \dots, x_K\}$ there exists $R > 0$ such that

$$\liminf_{m\to\infty} \Big(\sup_{t_m \le t \le t_m+\tau} E_f\big(u(t); B_R(x)\big)\Big) \le \liminf_{m\to\infty} E_f\big(u(t_m); B_{2R}(x)\big) + \frac{\varepsilon_1}{2} \le \varepsilon_1$$

where $\tau = \frac{\varepsilon_1 R^2}{2c_1 E(u_0)}$.

Furthermore, since K is bounded independently of the sequence $\{t_m\}$, we can repeatedly take subsequences to see that (for any such x) there exists $R > 0$ such that

$$\limsup_{m\to\infty} \Big(\sup_{t_m \le t \le t_m+\tau} E_f\big(u(t); B_R(x)\big)\Big) \le \varepsilon_1.$$

Now let $\Omega \subset\subset \mathcal{M} \setminus \{x_1, \dots, x_K\}$. Because of compactness, $\bar{\Omega}$ is covered by finitely many such balls $B_R(x)$. Hence

$$\int_{t_m}^{t_m+\tau}\int_\Omega |\nabla^2 u|^2 dx\ dt \le cE_f(u_0)\left(1+T+TR^{-2}\right),$$

by Lemma 3.3.4, and

$$\int_{t_m}^{t_m+\tau}\int_\Omega |\partial_t u|^2\ dx\ dt \to 0$$

by Lemma 3.3.2. Therefore, we may choose a new sequence $t'_m \in [t_m, t_m+\tau]$ such that $u_m = u(t'_m) \rightharpoonup u_\infty$ weakly in $W^{2,2}_{loc}\big(\mathcal{M}\setminus\{x_1,\dots,x_K\};\mathcal{N}\big)$, where $u_\infty : \mathcal{M}\setminus\{x_1,\dots,x_K\} \to \mathcal{N}$ is f-harmonic.

To finish, notice that by Theorem 2.1.5, u_∞ extends to a smooth f-harmonic map $u_\infty \in C^\infty(\mathcal{M},\mathcal{N})$. □

3.4.4 Singularities

Proposition 3.4.4. *Suppose $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$ and let u be a weak solution to (1.5.1) which is smooth on $\mathcal{M}\times(0,\infty)$ except at finitely many points. Then, at a singular point $(\bar{x},\bar{t})$, there exist sequences $x_m \to \bar{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ and a non-constant harmonic map $\tilde{u} : \mathbb{R}^2 \to \mathcal{N}$ with finite (harmonic) energy, such that as $m\to\infty$,*

$$u_m(x) := u\big(\exp_{x_m}(R_m x), t_m\big) \to \tilde{u} \tag{3.4.3}$$

in $W^{2,2}_{loc}(\mathbb{R}^2;\mathcal{N})$. Moreover $\tilde{u}$ extends to a smooth harmonic map $\bar{u} : \mathbb{R}^2 \cup \{\infty\} = S^2 \to \mathcal{N}$.

Proof.

Suppose that $(\bar{x},\bar{t})$ is singular. That is, for any $R \in (0, \frac{1}{2}\imath_{\mathcal{M}})$, there holds

$$\begin{cases} \limsup_{t\nearrow\bar{t}} E_f\big(u(t);B_R(\bar{x})\big) \ge \varepsilon_1 & \text{if } \bar{t}<\infty,\\ \liminf_{m\to\infty} E_f\big(u(t_m);B_R(\bar{x})\big) \ge \frac{1}{2}\varepsilon_1 & \text{if } \bar{t}=\infty,\end{cases}$$

where $\{t_m\}$ is as described by Proposition 3.4.3.

As the set of singular points is finite, $\bar{x}$ is isolated. Thus as $R_m \to 0$, we may pick $\bar{x}_m \to \bar{x}$ and $\bar{t}_m \nearrow \bar{t}$ such that

$$E_f\big(u(\bar{t}_m);B_{R_m}(\bar{x}_m)\big) = \sup_{x\in B_{2R_0}(\bar{x}),\bar{t}_m-\tau_m\le t\le\bar{t}_m} E_f\big(u(t);B_{R_m}(x)\big) = \frac{\varepsilon_1}{4},$$

for some $R_0>0$, where $\tau_m = \frac{\varepsilon_1 R_m^2}{16c_1E(u_0)}$. Without loss of generality, we assume that $\bar{x}_m \in B_{R_0}(\bar{x})$. We now rescale

$$\begin{aligned} w_m(x,t) &:= u\big(\bar{x}_m + R_m x, \bar{t}_m + R_m^2 t\big), \\ f_m(x) &:= f(\bar{x}_m + R_m x). \end{aligned}$$

Then w_m maps $B_{R_0/R_m} \times [t_0, 0] \to \mathcal{N}$, where $t_0 = -\frac{\varepsilon_1}{16 c_1 E(u_0)}$, and satisfies

$$\begin{aligned} \partial_t w_m(x,t) &= R_m^2 \big(\partial_t u\big)\big(\bar{x}_m + R_m x, \bar{t}_m + R_m^2 t\big) \\ \Delta w_m(x,t) &= R_m^2 \big(\Delta u\big)\big(\bar{x}_m + R_m x, \bar{t}_m + R_m^2 t\big) \\ A(w_m)(\nabla w_m, \nabla w_m) &= A(u)(R_m \nabla u, R_m \nabla u) \\ \nabla f_m(x) * \nabla w_m(x,t) &= R_m \nabla f(\bar{x}_m + R_m x) * R_m \nabla u\big(\bar{x}_m + R_m x, \bar{t}_m + R_m^2 t\big). \end{aligned}$$

Thus w_m satisfies

$$\partial_t w_m - f_m \Delta w_m = f_m A(w_m)(\nabla w_m, \nabla w_m) + \nabla f_m * \nabla w_m. \tag{3.4.4}$$

Furthermore

$$E_{f_m}\big(w_m(0), B_1\big) = \sup_{R_m \le |x| \le R_0, t_0 \le t \le 0} E_{f_m}\big(w_m(t); B_1(x)\big) = \frac{\varepsilon_1}{4},$$

$$\int_{t_0}^0 \int_{B_{R_0/R_m}} |\partial_t w_m|^2 \, dx \, dt \le \int_{\bar{t}_m - \tau_m}^{\bar{t}_m} \int_{\mathcal{M}} |\partial_t u|^2 \, dx \, dt \to 0$$

as $m \to \infty$. So by Lemma 3.3.4,

$$\int_{t_0}^0 \int_{B_{R_0/2R_m}} |\nabla^2 w_m|^2 \, dx \, dt \le c$$

uniformly. Thus we may pick a sequence $s_m \in [t_0, 0]$ and set $\tilde{w}_m(x) := w_m(x, s_m)$ such that $\partial_t w_m(\cdot, s_m) \to 0$ in L^2 and $\tilde{w}_m \rightharpoonup \tilde{u}$ weakly in $W^{2,2}_{loc}(\mathbb{R}^2; \mathcal{N})$ as well as $\tilde{w}_m \to \tilde{u}$ strongly in $W^{1,2}_{loc}(\mathbb{R}^2; \mathcal{N})$. (We could use the same method used earlier to show also that $\tilde{w}_m \to \tilde{u}$ strongly in $W^{2,2}_{loc}(\mathbb{R}^2; \mathcal{N})$.)

Passing to the limit in (3.4.4), we see that $\tilde{u}$ is harmonic (note that $\tilde{u}$ is 1-harmonic, not f-harmonic). We also see that for large m,

$$\begin{aligned} f(\bar{x}) E_1\big(\tilde{u}; B_2\big) &= E_{f_m}\big(w_m(s_m), B_2\big) - o(1) \\ &\ge E_{f_m}\big(w_m(0), B_1\big) - c_1 s_m E_f(u_0) - o(1) \\ &\ge \frac{\varepsilon_1}{4} - \frac{\varepsilon_1}{16} - o(1) > 0 \end{aligned}$$

where $o(1) \to 0$ as $m \to \infty$. Therefore $\tilde{u}$ is non-constant.

Now $E_1(\tilde{u}) \leq c \liminf_{m\to\infty} E_{f_m}(\tilde{w}_m) \leq cE_f(u_0)$, so by [SU81, Theorem 3.6], $\tilde{u}$ extends to a harmonic map $\bar{u} : S^2 = \mathbb{R}^2 \cup \{\infty\} \to \mathcal{N}$.

□

3.4.5 Uniqueness

Proposition 3.4.5. *If u, v are solutions to (1.5.1) with $\partial_t u$, $\partial_t v$, $\nabla^2 u$, $\nabla^2 v \in L^2(\mathcal{M} \times [0,T])$ and $u|_{t=0} = v|_{t=0} = u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$, then $u \equiv v$.*

Proof.
Set $w = u - v$. By (1.5.1) and (3.3.1)

$$\begin{aligned}\big|\partial_t w - f\Delta w\big| &= \Big|fA(u)(\nabla u, \nabla u) + \nabla f * \nabla u - fA(v)(\nabla v, \nabla v) - \nabla f * \nabla v\Big| \\ &\leq c|w|\big(|\nabla u|^2 + |\nabla v|^2 + |\nabla u| + |\nabla v|\big) + c|\nabla w|\big(|\nabla u| + |\nabla v| + 1\big).\end{aligned}$$

Thus

$$\begin{aligned}\int_{\mathcal{M}} |w(t_0)|^2 \, dx + 2\int_0^{t_0}\int_{\mathcal{M}} |\nabla w|^2 \, dx \, dt \\ &\leq c\int_0^{t_0}\int_{\mathcal{M}} |w||\partial_t w - f\Delta w| \, dx \, dt + c\int_0^{t_0}\int_{\mathcal{M}} |w||\nabla w| \, dx \, dt \\ &\leq c\int_0^{t_0}\int_{\mathcal{M}} |w|^2\big(|\nabla u|^2 + |\nabla v|^2 + |\nabla u| + |\nabla v|\big) \, dx \, dt \\ &\quad + c\int_0^{t_0}\int_{\mathcal{M}} |w||\nabla w|\big(|\nabla u| + |\nabla v| + 1\big) \, dx \, dt \\ &\leq c\int_0^{t_0}\int_{\mathcal{M}} |w|^2\big(|\nabla u|^2 + |\nabla v|^2 + |\nabla u| + |\nabla v| + 1\big) \, dx \, dt \\ &\quad + 1\int_0^{t_0}\int_{\mathcal{M}} |\nabla w|^2 \, dx \, dt\end{aligned} \tag{3.4.5}$$

where we have again used $2ab \leq \delta a^2 + \delta^{-1}b^2$ as well as $w|_{t=0} = 0$.

By Lemma 3.3.1 (and as $\mathcal{M}$ is compact),

$$\begin{aligned}\int_0^{t_0}\int_{\mathcal{M}} \big(|\nabla u|^4 + |\nabla v|^4 + |\nabla u|^2 + |\nabla v|^2 + 1\big) \, dx \, dt \\ &\leq cE_f(u_0)\int_0^{t_0}\int_{\mathcal{M}} \big(|\nabla^2 u|^2 + |\nabla^2 v|^2\big) \, dx \, dt \\ &\quad + ct_0\big(E_f(u_0)\big)^2 + ct_0 E_f(u_0) + ct_0 \\ &\leq \delta^2\end{aligned}$$

if $0 < t_0 \le T_0$ and if $T_0 = T_0(\delta) \in (0,1)$ is sufficiently small; and

$$\begin{aligned}
&\int_0^{t_0}\int_{\mathcal{M}} |w|^4 \, dx \, dt \\
&\qquad \le c\Big(\sup_{0\le t\le t_0}\int_{\mathcal{M}} |w(t)|^2 \, dx\Big)\Big(\int_0^{t_0}\int_{\mathcal{M}} |\nabla w|^2 \, dx \, dt + \int_0^{t_0}\int_{\mathcal{M}} |w|^2 \, dx \, dt\Big) \\
&\qquad \le c\Big((1+t_0)\sup_{0\le t\le t_0}\int_{\mathcal{M}} |w(t)|^2 \, dx + \int_0^{t_0}\int_{\mathcal{M}} |\nabla w|^2 \, dx \, dt\Big)^2.
\end{aligned}$$

Now suppose that $t_1 \le T_0$. We choose $t_0 \in [0, t_1]$ such that

$$\int_{\mathcal{M}} |w(t_0)|^2 \, dx = \sup_{0\le t\le t_1}\int_{\mathcal{M}} |w(t)|^2 \, dx = \sup_{0\le t\le t_0}\int_{\mathcal{M}} |w(t)|^2 \, dx.$$

Then we see from (3.4.5) that

$$\begin{aligned}
&\sup_{0\le t\le t_0}\int_{\mathcal{M}} |w(t)|^2 \, dx + \int_0^{t_0}\int_{\mathcal{M}} |\nabla w|^2 \, dx \, dt \\
&\le c\Big(\int_0^{t_0}\int_{\mathcal{M}} |w|^4 \, dx \, dt\Big)^{\frac{1}{2}}\Big(\int_0^{t_0}\int_{\mathcal{M}} \big(|\nabla u|^4 + |\nabla v|^4 + |\nabla u|^2 + |\nabla v|^2 + 1\big) \, dx \, dt\Big)^{\frac{1}{2}} \\
&\le c\delta\Big((1+t_0)\sup_{0\le t\le t_0}\int_{\mathcal{M}} |w(t)|^2 \, dx + \int_0^{t_0}\int_{\mathcal{M}} |\nabla w|^2 \, dx \, dt\Big),
\end{aligned}$$

for $c = c(\mathcal{M}, \mathcal{N}, f)$. If we now choose $\delta > 0$ such that $c\delta < \frac{1}{2}$, then we see that

$$\sup_{0\le t\le t_0}\int_{\mathcal{M}} |w(t)|^2 \, dx + \int_0^{t_0}\int_{\mathcal{M}} |\nabla w|^2 \, dx \, dt \quad \le T_0 \sup_{0\le t\le t_0}\int_{\mathcal{M}} |w(t)|^2 \, dx,$$

and as $T_0 \le 1$, it follows that $w \equiv 0$ on $\mathcal{M} \times [0, T_0(\delta)]$.

Therefore the maximum interval $I \subset [0,T]$ containing $t = 0$ and such that $u(t) = v(t)$ for $t \in I$, is relatively open. But I is also closed, so we have uniqueness. □

This completes the proof of Theorem 3.1.1.

Chapter 4

Mapping the boundary to a point

4.1 Analogue of a theorem by Lemaire

The reader is asked now to recall a theorem of Lemaire [Lem78, Theorem 3.2], regarding harmonic maps, which states: "*Let $\mathcal{M}$ be a compact contractible surface with boundary, and let p be a point in $\mathcal{N}$. Every harmonic map $\mathcal{M} \to \mathcal{N}$ which maps $\partial\mathcal{M}$ onto p is constant, and takes value p.*" As we will see in the following section, the direct analogue involving f-harmonic maps is not true; it is for example, possible to find a non-trivial f-harmonic map $D \to S^2$ which maps ∂D to a point.

We do however find a partial analogue of the quoted result – if we place suitable restrictions on the f. Presented here is a simple demonstration of a restriction applied to the f, which denies the existence of any non-constant f-harmonic maps $D \to S^2$.

Proposition 4.1.1. *Suppose that $f : D \to (0, \infty)$ satisfies $\nabla f(x) \cdot x > 0$ for almost all $x \in D$. Then every smooth f-harmonic map $u \in C^\infty(D; \mathcal{N})$ which maps ∂D to a point p, is constant and takes the value p.*

The strategy for the proof is as follows: Assuming that there is a non-constant f-harmonic map, for such an f, we pre-compose u with a particular rotationally symmetric variation $D \to D$ (see figure 4.1 on page 51). The purpose of this is to "squash" the energy away from the boundary (high f) towards the origin (low f). This "should" decrease the overall energy, hence proving that u cannot be f-harmonic. However, the distortion on an annulus close to the boundary (i.e. $D_b \setminus D_a$ for $a < b$ both close to 1) "may" add enough to the f-energy to cancel out the decrease elsewhere. Fortunately, we are able to rule this possibility out by studying the Hopf Differential. The following technical lemma gives this calculation.

Lemma 4.1.2. *Let $u \in C^\infty(D, \mathcal{N})$ be an f-harmonic map which maps ∂D to a point. Then, for $0 < a < b < 1$ we have that*

$$\begin{aligned} &-\left(\frac{b}{b-a}\right)\int_{D_b\setminus D_a} \frac{f}{r}\Big(\left|u_r\right|^2 - \frac{1}{r^2}\left|u_\phi\right|^2\Big)\ dxdy \\ &\le \frac{1}{a}\Bigg\{ -\int_0^{2\pi} |u_r|^2\ d\phi\Bigg|_{|z|=1} + 6\left\|\nabla f\right\|_{L^\infty}\int_{D\setminus D_a} |\nabla u|^2\ dxdy\Bigg\}. \end{aligned} \tag{4.1.1}$$

Proof.

Consider the Hopf Differential ψdz^2, where $\psi(u) = |u_x|^2 - |u_y|^2 - 2i\left\langle u_x, u_y\right\rangle$ in cartesian coordinates and $\psi(u) = \frac{\bar{z}^2}{r^2}\left(|u_r|^2 - \frac{1}{r^2}|u_\phi|^2 - \frac{2i}{r}\left\langle u_r, u_\phi\right\rangle\right)$ in polar coordinates. It is well known that if u is harmonic then $\bar{\partial}\psi = 0$. For u f-harmonic, we calculate (see margin notes) that

$z = x + iy$
$x = r\cos\phi$
$y = r\sin\phi$

$r_x = \frac{x}{r}$
$r_y = \frac{y}{r}$
$\phi_x = \frac{-y}{r^2}$
$\phi_y = \frac{x}{r^2}$

$$\begin{aligned} \bar{\partial}\psi(u) :&= \frac{1}{2}\big(\psi_x + i\psi_y\big) = \left\langle u_x - iu_y, \tau(u)\right\rangle \\ &= \frac{1}{f}\left\langle \bar{z}\left(\frac{u_r}{r} - i\frac{u_\phi}{r^2}\right), f_r u_r + \frac{1}{r^2}f_\phi u_\phi\right\rangle \\ &= \left(\frac{\bar{z}}{rf}\right)\left(\left[f_r|u_r|^2 + \frac{1}{r^2}f_\phi\left\langle u_r, u_\phi\right\rangle\right] - \frac{i}{r}\left[f_r\left\langle u_r, u_\phi\right\rangle + \frac{1}{r^2}f_\phi|u_\phi|^2\right]\right) \end{aligned} \tag{4.1.2}$$

and that

$x_r = x/r$
$x_\phi = -y$
$y_r = y/r$
$y_\phi = x$

$$\begin{aligned} \mathrm{Re}\left[(f_x + if_y)\psi z\right] &= \mathrm{Re}\left(\frac{1}{r}f_r + \frac{i}{r^2}f_\phi\right)\frac{z^2\bar{z}^2}{r^2}\left(|u_r|^2 - \frac{1}{r^2}|u_\phi|^2 - \frac{2i}{r}\left\langle u_r, u_\phi\right\rangle\right) \\ &= rf_r\left(|u_r|^2 - \frac{1}{r^2}|u_\phi|^2\right) + 2\frac{1}{r}f_\phi\left\langle u_r, u_\phi\right\rangle. \end{aligned} \tag{4.1.3}$$

Notice that by Cauchy-Stokes

$$\begin{aligned} \int_{|z|=1} f\psi(z)z\ dz - \int_{|z|=r} f\psi(z)z\ dz = &\int_{D\setminus D_r} f\big(\bar{\partial}\psi\big)z\ d\bar{z}\wedge dz \\ &+ \frac{1}{2}\int_{D\setminus D_r}(f_x + if_y)\psi(z)z\ d\bar{z}\wedge dz. \end{aligned} \tag{4.1.4}$$

We also calculate that

$$\begin{aligned}
\operatorname{Re}\psi z^2 &= \operatorname{Re}\left(|u_x|^2 - |u_y|^2 - 2i\left\langle u_x, u_y\right\rangle\right)\left(x^2 - y^2 + 2ixy\right) \\
&= \left(x^2 - y^2\right)\left(|u_x|^2 - |u_y|^2\right) + 4xy\left\langle u_x, u_y\right\rangle \\
&= \Big[|u_x|^2 x^2 + |u_y|^2 y^2 + 2xy\left\langle u_x, u_y\right\rangle\Big] \\
&\qquad - \Big[|u_x|^2 y^2 + |u_y|^2 x^2 - 2xy\left\langle u_x, u_y\right\rangle\Big] \\
&= \left|u_x x + u_y y\right|^2 - \left|u_x y - u_y x\right|^2 \\
&= r^2\left|u_x x_r + u_y y_r\right|^2 - \left|u_x x_\phi + u_y y_\phi\right|^2 \\
&= |z|^2\left|u_r\right|^2 - \left|u_\phi\right|^2.
\end{aligned} \tag{4.1.5}$$

Therefore,

$$\begin{aligned}
\int_0^{2\pi} f\left[r^2\left|u_r\right|^2 - \left|u_\phi\right|^2\right]\, d\phi\bigg|_{|z|=r} &= \operatorname{Re}\int_0^{2\pi} f\psi(z)z^2\, d\phi\bigg|_{|z|=r} \\
&= \operatorname{Im}\int_{|z|=r} f\psi(z)z\, dz.
\end{aligned} \tag{4.1.6}$$

In the case of u harmonic (i.e. $f \equiv 1$), we could use $\operatorname{Im}\int_{|z|=r}\psi(z)z\, dz = 0$ to obtain a stronger result. For u f-harmonic, we instead calculate

$$\begin{aligned}
Q :&= -\int_{D_b\setminus D_a} \frac{f}{r}\left[\left|u_r\right|^2 - \frac{1}{r^2}\left|u_\phi\right|^2\right]\, dxdy \\
&= -\int_a^b \frac{1}{r^2}\int_0^{2\pi} f\left[r^2\left|u_r\right|^2 - \left|u_\phi\right|^2\right]\, d\phi\, dr \\
&= -\int_a^b \frac{1}{r^2}\left[\operatorname{Im}\int_{|z|=r} f\psi(z)z\, dz\right]\, dr
\end{aligned}$$

by (4.1.6). It follows that

$$
\begin{aligned}
Q = &-\int_a^b \frac{1}{r^2}\,\mathrm{Im}\Big[\int_{|z|=1}\psi(z)z\;dz - \int_{D\setminus D_r} f\big(\bar\partial\psi\big)z\;d\bar z\wedge dz \\
&- \int_{D\setminus D_r}\frac{1}{2}(f_x+if_y)\psi z\;d\bar z\wedge dz\Big]\;dr \\
= &-\int_a^b \frac{1}{r^2}\;dr\,\mathrm{Re}\left[\int_0^{2\pi}\psi(z)z^2\;d\phi\right]_{|z|=1} \\
&+\int_a^b \frac{1}{r^2}\,\mathrm{Im}\left[\int_{D\setminus D_r} f\big(\bar\partial\psi\big)z2i\;dx\wedge dy\right]\;dr \\
&+\int_a^b \frac{1}{r^2}\,\mathrm{Im}\left[\int_{D\setminus D_r}\frac{1}{2}(f_x+if_y)\psi z2i\;dx\wedge dy\right]\;dr
\end{aligned}
$$

by (4.1.4) and then (4.1.6) again. Here we have used $d\bar z\wedge dz = 2idx\wedge dy$. Then by (4.1.2), (4.1.3) and (4.1.5), and since $u_\phi|_{|z|=1}=0$, we see that

$$
\begin{aligned}
Q = &-\int_a^b \frac{1}{r^2}\;dr\left[\int_0^{2\pi}|u_r|^2\;d\phi\right]_{|z|=1} \\
&+\int_a^b \frac{2}{r^2}\int_{D\setminus D_r}|z|\left[f_r|u_r|^2+\frac{1}{|z|^2}f_\phi\left\langle u_r,u_\phi\right\rangle\right]\;dx\wedge dy\;dr \\
&+\int_a^b \frac{1}{r^2}\int_{D\setminus D_r}\left[|z|f_r\left(|u_r|^2-\frac{1}{|z|^2}|u_\phi|^2\right)+\frac{2}{|z|}f_\phi\left\langle u_r,u_\phi\right\rangle\right]\;dx\wedge dy\;dr \\
= &-\int_a^b \frac{1}{r^2}\;dr\left[\int_0^{2\pi}|u_r|^2\;d\phi\right]_{|z|=1} \\
&+\int_a^b \frac{1}{r^2}\int_{D\setminus D_r}\left[|z|f_r\left(3|u_r|^2-\frac{1}{|z|^2}|u_\phi|^2\right)+\frac{4}{|z|}f_\phi\left\langle u_r,u_\phi\right\rangle\right]\;dx\wedge dy\;dr.
\end{aligned}
$$

Finally, we estimate the second integral to see

$$
\begin{aligned}
Q &\le -\left(\frac{b-a}{ab}\right)\left[\int_0^{2\pi}|u_r|^2\;d\phi\right]_{|z|=1}+\left(\frac{b-a}{ab}\right)\int_{D\setminus D_a}3|z|\left(|f_r|+\frac{|f_\phi|}{|z|}\right)|\nabla u|^2\;dx\wedge dy \\
&\le -\left(\frac{b-a}{ab}\right)\left[\int_0^{2\pi}|u_r|^2\;d\phi\right]_{|z|=1}+6\left\|\nabla f\right\|_{L^\infty}\left(\frac{b-a}{ab}\right)\int_{D\setminus D_a}|\nabla u|^2\;dx\wedge dy.
\end{aligned}
$$

□

Proof of Proposition 4.1.1.

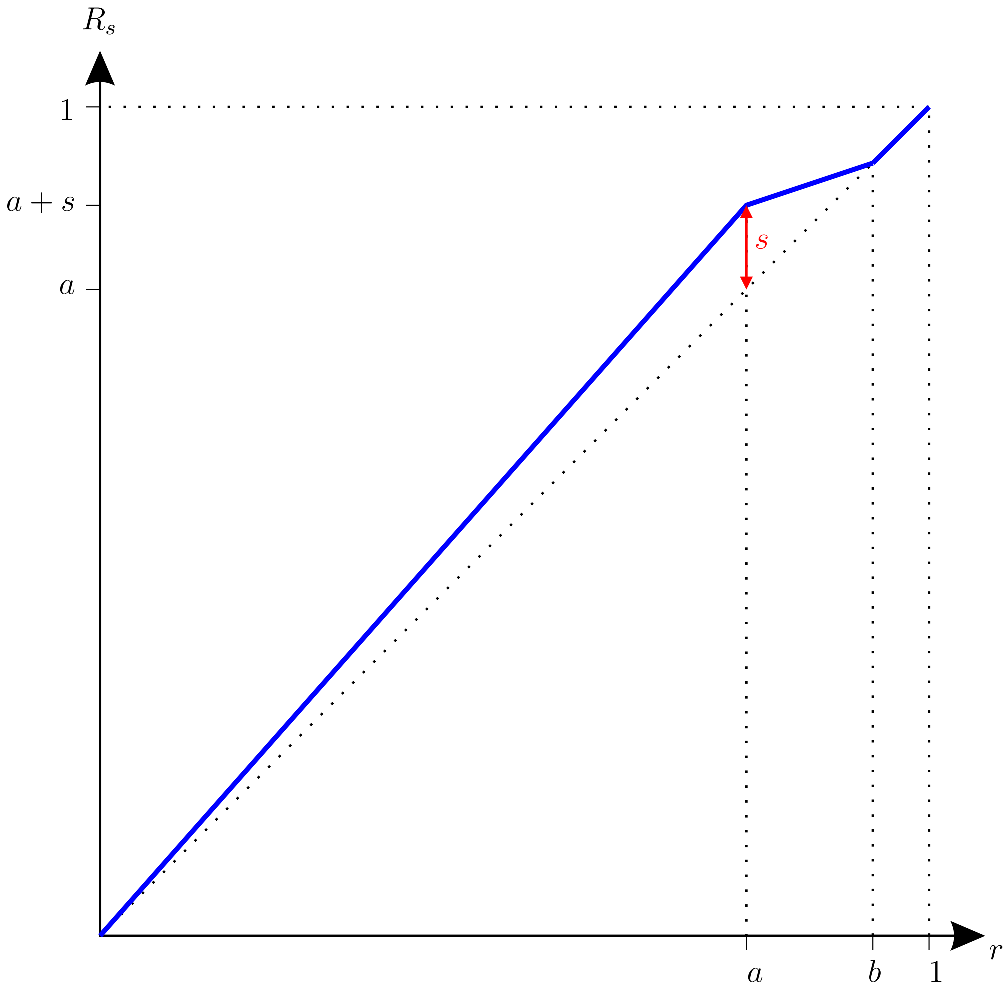

Figure 4.1: The map $R_s : [0,1] \to [0,1]$.

Suppose that $u : D \to \mathcal{N}$ is a non-constant f-harmonic map, mapping ∂D to a point. Define $R_s : [0,1] \to [0,1]$ by (see figure 4.1)

$$R_s = \begin{cases} (1+\frac{s}{a})r & r \in [0,a), \\ (r-a)(1-\frac{s}{b-a}) + a + s & r \in [a,b), \\ r & r \in [b,1]. \end{cases} \tag{4.1.7}$$

The reader should imagine here that $(1-a)$ is small. Consider the variation $u_s(x) := u(y_s)$ where in polar coordinates $x = (r,\phi)$ and $y_s = (R_s,\phi)$. We will omit the "$_s$" notation on y_s and R_s.

Recall that while this variation is not admissible (Definition 1.1.3 on page 1), this does not cause us a problem by Lemma 1.3.5 on page 7.

Suppose now (until after (4.1.8)) that $r \in [a, b)$. Then

$$r = \frac{R - \frac{sb}{b-a}}{1 - \frac{s}{b-a}}.$$

So

$$\frac{dr}{dR} = \frac{1}{1 - \frac{s}{b-a}}$$

and the volume elements dx and dy satisfy

$$dx = r dr d\phi = r\frac{dr}{dR} dR d\phi = \frac{R - \frac{sb}{b-a}}{(1 - \frac{s}{b-a})^2} dR d\phi = \frac{1}{R}\frac{R - \frac{sb}{b-a}}{(1 - \frac{s}{b-a})^2} dy.$$

Moreover

$$\begin{aligned}
\nabla u_s(x) &= \hat{\boldsymbol{r}}\frac{\partial u_s}{\partial r}(x) + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial u_s}{\partial \phi}(x) \\
&= \hat{\boldsymbol{r}}\frac{\partial u}{\partial R}(y)\frac{\partial R}{\partial r}(x) + \frac{1}{r}\hat{\boldsymbol{\phi}}\frac{\partial u}{\partial \phi}(y) \\
&= \hat{\boldsymbol{r}}\frac{\partial u}{\partial R}(y)\left(1 - \frac{s}{b-a}\right) + \frac{1 - \frac{s}{b-a}}{R - \frac{sb}{b-a}}\hat{\boldsymbol{\phi}}\frac{\partial u}{\partial \phi}(y) \\
&= R\frac{1 - \frac{s}{b-a}}{R - \frac{sb}{b-a}}\left[\frac{R - \frac{sb}{b-a}}{R}\hat{\boldsymbol{R}}\frac{\partial u}{\partial R} + \frac{1}{R}\hat{\boldsymbol{\phi}}\frac{\partial u}{\partial \phi}\right](y) \\
&= R\frac{1 - \frac{s}{b-a}}{R - \frac{sb}{b-a}}\left[\nabla u - \frac{sb}{R(b-a)}\hat{\boldsymbol{R}}\frac{\partial u}{\partial R}\right](y)
\end{aligned}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors in the directions of increasing r and ϕ respectively. So

$$|\nabla u_s(x)|^2 = R^2\left(\frac{1 - \frac{s}{b-a}}{R - \frac{sb}{b-a}}\right)^2\left[|\nabla u|^2 - \frac{2sb}{R(b-a)}\left|\frac{\partial u}{\partial R}\right|^2 + \left(\frac{sb}{R(b-a)}\right)^2\left|\frac{\partial u}{\partial R}\right|^2\right](y). \tag{4.1.8}$$

We may then calculate that

$$
\begin{aligned}
E_f(u_s) &= \frac{1}{2}\int_{D_a} f(x)\,|\nabla u_s(x)|^2\,dx \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a} f(x)\,|\nabla u_s(x)|^2\,dx \\
&\quad + E_f(u_s; D\setminus D_b) \\
&= \frac{1}{2}\int_{D_{a+s}} f\left(y\left(\frac{1}{1+\frac{s}{a}}\right)\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_{a+s}} f\left(y\frac{1}{|y|}\left(\frac{|y|-\frac{sb}{b-a}}{1-\frac{s}{b-a}}\right)\right)\frac{R}{R-\frac{sb}{b-a}}\Big[\cdots\Big]\,dy \\
&\quad + E_f(u; D\setminus D_b)
\end{aligned}
$$

where the notation "$\big[\cdots\big]$" refers to the contents of the square parentheses in (4.1.8). We may then calculate

$$
\begin{aligned}
\frac{d}{ds}E_f(u_s)\bigg|_{s=0} &= \frac{1}{2}\int_{D_a}\nabla f\cdot y\left(\frac{-1}{a}\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a}\nabla f\cdot\frac{y}{|y|}\left(\frac{|y|-b}{b-a}\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a} f(y)\frac{1}{R}\left(\frac{b}{b-a}\right)|\nabla u|^2\,dy \\
&\quad + \frac{1}{2}\int_{D_b\setminus D_a} f(y)\left(\frac{-2b}{R(b-a)}\right)\left|\frac{\partial u}{\partial R}\right|^2\,dy \\
&= -\frac{1}{2a}\int_{D_a}\nabla f\cdot y|\nabla u|^2\,dy \\
&\quad - \frac{1}{2}\int_{D_b\setminus D_a}\nabla f\cdot\frac{y}{|y|}\left(\frac{b-|y|}{b-a}\right)|\nabla u|^2\,dy \\
&\quad - \frac{1}{2}\int_{D_b\setminus D_a} f(y)\left(\frac{b}{b-a}\right)\frac{1}{R}\left[\left|\frac{\partial u}{\partial R}\right|^2-\frac{1}{R^2}\left|\frac{\partial u}{\partial \phi}\right|^2\right]\,dy.
\end{aligned}
$$

The reader should notice that the two boundary derivatives – i.e. the two integrals over $|z| = a + s$ – cancel when $s = 0$.

It follows by Lemma 4.1.2 that

$$\begin{aligned}\frac{d}{ds}E_f(u_s)\bigg|_{s=0} &\leq -\frac{1}{2a}\int_{D_a}\nabla f\cdot y|\nabla u|^2\,dy \\ &\quad -\frac{1}{2}\int_{D_b\setminus D_a}\nabla f\cdot\frac{y}{|y|}\left(\frac{b-|y|}{b-a}\right)|\nabla u|^2\,dy \\ &\quad -\frac{1}{2a}\left[\int_0^{2\pi}|u_r|^2\,d\phi\right]_{|z|=1} \\ &\quad +\frac{3}{a}\|\nabla f\|_{L^\infty}\int_{D\setminus D_a}|\nabla u|^2\,dy \\ &< 0\end{aligned} \tag{4.1.9}$$

for a sufficiently close to 1, contradicting u being f-harmonic. □

Remark. Alternately, we could base the above proof on the variation (based on the map $[0,1]\to[0,1]$);

$$R'_s = \begin{cases}(\frac{a}{a-s})r & r\in[0,a-s),\\ \frac{b-a}{b-a+s}(r-a+s)+a & r\in[a-s,b),\\ r & r\in[b,1].\end{cases} \tag{4.1.10}$$

See figure 4.2 on page 55. This would slightly simplify the calculation in places, while making a little extra work elsewhere.

Remark. The previous lemma has the hypothesis $\nabla f(x)\cdot x > 0$. One would expect this result to also hold with an alternate hypothesis of f being a convex function – that is if for all $x, y\in D$, $x\neq y$, and all $\lambda\in(0,1)$ there holds

$$f(\lambda x+(1-\lambda)y) < \lambda f(x)+(1-\lambda)f(y).$$

Certainly if f has minimum at x_0 in the interior of D then $\nabla f(x)\cdot(x-x_0) > 0$ and so such a proof should be possible by distorting about x_0, instead of about 0 as we did above.

Remark. One would also expect the result to hold for the weaker hypothesis $\nabla f\cdot x\geq 0$. If f is constant then we are just in the harmonic case. Alternately there must exist $z_0\in D$ such that $\nabla f(z_0)\cdot z_0 > 0$. But then $\nabla f\cdot x > 0$ on some neighbourhood Ω of z_0 and the above argument shows that u must be constant on Ω. Some kind of unique continuation argument should show that u must be constant on D.

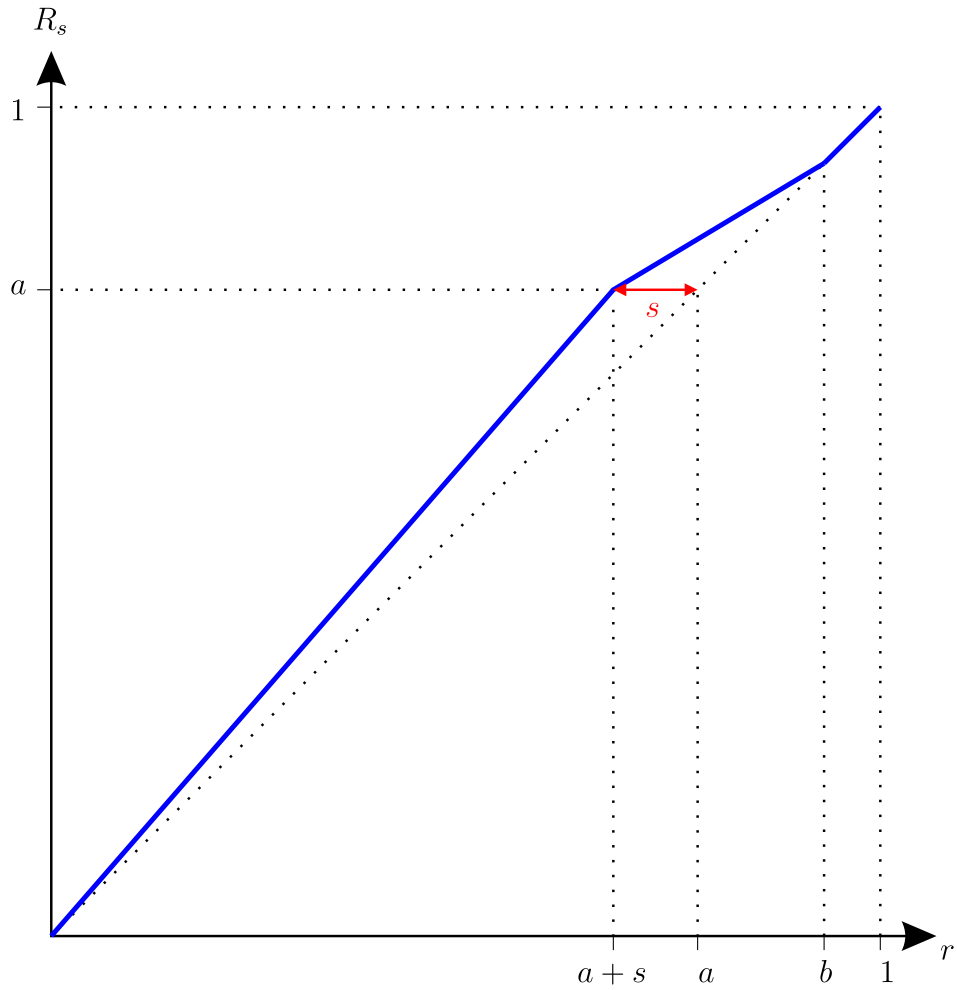

Figure 4.2: The map $R_s' : [0,1] \to [0,1]$.

4.2 The existence of non-trivial f-harmonic maps $D \to S^2$ $(\partial D \mapsto \{0\})$ and $T^2 \to S^2$

We present a pair of results. As their proofs are very similar, we offer here only one for the latter.

Lemma 4.2.1. *There exist an f and a smooth non-constant, f-harmonic map from the disc to the 2-sphere which maps ∂D to a point.*

There is also a result by Eells-Wood [EW82] which states that: *There*

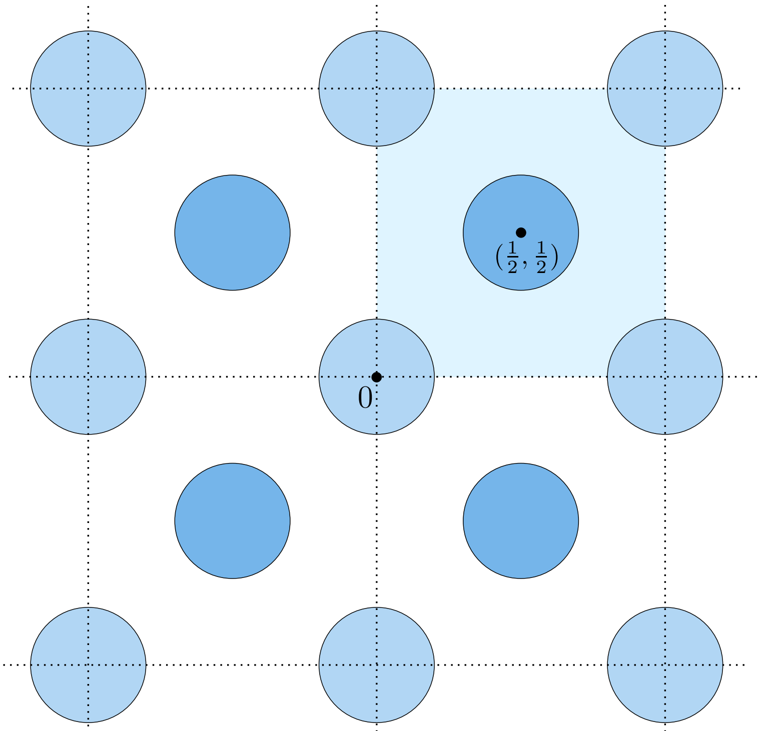

Figure 4.3: The flat-square torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Shown are the small balls $B_\delta(0)$ and $B_\delta(\frac{1}{2}, \frac{1}{2})$.

does not exist a harmonic map of degree 1, from the torus to the 2*-sphere.* However:

Lemma 4.2.2. *There exist $f : T^2 \to (0, \infty)$ and a smooth degree* 1*, f-harmonic map from the square torus to the 2-sphere.*

Proof of Lemma 4.2.2.

Consider $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\delta \in (0, \frac{1}{100})$ be chosen later. Take an f with $f \equiv 1$ on $T^2 \setminus [B_\delta(0) \cup B_\delta(\frac{1}{2}, \frac{1}{2})]$, $f(0) = f(\frac{1}{2}, \frac{1}{2}) = 2$, and $1 < f(x) < 2$ otherwise. Suppose further that f is invariant under isometries $T^2 \to T^2$ which fix 0. Such isometries form a subset Φ, of the set of all isometries $\mathbb{R}^2 \to \mathbb{R}^2$. So Φ must be $\{e, R, R^2, R^3, r_{x=\frac{1}{2}}, r_{y=\frac{1}{2}}, r_{x=y}, r_{x=-y}\}$, where e, R and $r_{x=y}$ denote respectively; the identity, anticlockwise rotation by $\frac{1}{2}\pi$ about $(\frac{1}{2}, \frac{1}{2})$ (or equivalently about $(0, 0)$), and reflection in the line $x = y$. Note that the only points fixed by every element of Φ are 0 and $(\frac{1}{2}, \frac{1}{2})$.

Now consider an initial map $u_0 : T^2 \to S^2$ which, for small ε, maps $T^2 \setminus B_\varepsilon(0)$ onto a small neighbourhood of the "south pole" and maps $B_\varepsilon(0)$ once around the remainder of S^2. So u_0 is of degree 1. We suppose further that u_0 has the same symmetry as f, i.e. that it is unchanged by isometries in Φ. It is known that we may do this in such a way so that the (1-) harmonic energy satisfies $E_1(u_0) < 6\pi$. Now choose δ sufficiently small so that we have $E_f(u_0) < 7\pi$.

We now study the heat flow $u : T^2 \times [0, \infty) \to S^2$,

$$\begin{cases} u_t - f(x)\Delta u = u|\nabla u|^2 + \nabla f * \nabla u \\ u|_{t=0} = u_0. \end{cases} \tag{4.2.1}$$

We know from the f-harmonic Heat Flow Theorem (Theorem 3.1.1), that away from bubble points $(z_0, t_0) \in T^2 \times (0, \infty]$, the map u is smooth and $u(\cdot, t_n)$ converges smoothly to an f-harmonic map, u_∞ say, as $t_n \to \infty$ (for some suitable sequence t_n as asserted by Theorem 3.1.1). If we can show that there can be no bubbling in this flow, then we would have the existence of a degree 1, f-harmonic map $T^2 \to S^2$.

Because $E_f(u_0) < 7\pi$, and because the "energy lost in a bubble" at z_0 is $4\pi f(z_0) \geq 4\pi$ (if $f \equiv 1$, it is well known that the energy lost due to bubbling would be a multiple of 4π), there can be at most one bubble in this heat flow. Now suppose that a bubble does indeed form. Suppose a bubble forms at the point and time (z_0, t_0). By z_0, we mean $z_0 + \mathbb{Z}^2$ where $z_0 \in [0, 1)^2$. Because of the symmetry that we started with, if we have a bubble at z_0, then we must also have one at $I(z_0)$ for any isometry $I \in \Phi$. As noted previously, the only points fixed under such isometries are 0 and $(\frac{1}{2}, \frac{1}{2})$, and because we can have at most one bubble, these are the only places where a bubble could possibly form.

However, the "energy lost" if a bubble formed at either one of these points would be $4\pi f(z_0) = 8\pi$ – greater then the amount of energy that we started with.

Therefore there can be no bubbling. So there exists a smooth degree 1, f-harmonic map $u_\infty : T^2 \to S^2$. □

4.3 An example of a non-constant f-harmonic map $D \to S^2$ which maps ∂D to a point.

Now that we know that there does exist a non-constant f-harmonic map $D \to S^2$, it is of interest to understand what such a map would look like, or more relevantly what the associated function f would look like. A guess, given Proposition 4.1.1, is that a convex f is no use to us here. In order to simplify our calculations we introduce so called "longitudinally symmetric maps".

We say that the function $f : D \to (0,\infty)$ is *rotationally symmetric* if we can write $f(x) = f_1(|x|)$ for some $f_1 : [0,1] \to (0,\infty)$. For $\alpha \in C^\infty([0,1],\mathbb{R})$ such that $\alpha(0) = 0$, define $U_\alpha : D \to S^2 \subset \mathbb{R}^3$ by

$$U_\alpha(x) = \left(\frac{x}{|x|} \sin\alpha(|x|), \cos\alpha(|x|) \right).$$

The map $u : D \to S^2$ is said to be *longitudinally symmetric* if $u(x) = U_\theta(x)$ for some $\theta : [0,1] \to \mathbb{R}$.

4.3.1 The f-harmonic equation for longitudinally symmetric maps $D \to S^2$ mapping ∂D to a point.

We start by calculating the Euler-Lagrange equation for a longitudinally symmetric f-harmonic map, assuming that f is rotationally symmetric. Throughout the rest of this section, we use polar coordinates (r,ϕ) and (ϕ,θ) on D and S^2 respectively.

Lemma 4.3.1. *Let $f : D \to (0,\infty)$ depend only on r. Suppose that the map $u : D \to S^2$ is of the form $(r,\phi) \mapsto (\phi,\theta(r))$ (i.e. u is longitudinally symmetric), that $\theta(0) = 0$ and that $\theta(1) = \pi$. Then u is f-harmonic if and only if*

$$r^2\theta_{rr} + \theta_r\left(r + r^2\frac{f_r}{f}\right) = \sin\theta\cos\theta. \tag{4.3.1}$$

Proof.

We know that u is f-harmonic if and only if

$$\Delta u + u|\nabla u|^2 + \frac{1}{f}\nabla f * \nabla u = 0 \tag{4.3.2}$$

by Lemma 1.3.1.

Using polar coordinates (ϕ,θ) on S^2, we have that $u_1(r,\phi) = \cos\phi\sin\theta(r)$, $u_2(r,\phi) = \sin\phi\sin\theta(r)$ and $u_3(r,\phi) = \cos\theta(r)$. Differentiating gives

$$\begin{aligned}
\nabla u_1 &= \theta_r\cos\phi\cos\theta\hat{\boldsymbol{r}} - \frac{1}{r}\sin\phi\sin\theta\hat{\boldsymbol{\phi}}, \\
\nabla u_2 &= \theta_r\sin\phi\cos\theta\hat{\boldsymbol{r}} + \frac{1}{r}\cos\phi\sin\theta\hat{\boldsymbol{\phi}}, \\
\nabla u_3 &= -\theta_r\sin\theta\hat{\boldsymbol{r}},
\end{aligned}$$

where $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ denote unit vectors in the domain D in the directions of increasing r and ϕ respectively. It follows that

$$|\nabla u|^2 = \theta_r^2 + \frac{1}{r^2}\sin^2\theta. \tag{4.3.3}$$

Moreover $\nabla f = f_r \hat{\boldsymbol{r}}$, so we see that

$$\begin{aligned} \nabla f \cdot \nabla u_1 &= f_r \theta_r \cos\phi \cos\theta, \\ \nabla f \cdot \nabla u_2 &= f_r \theta_r \sin\phi \cos\theta, \\ \nabla f \cdot \nabla u_3 &= -f_r \theta_r \sin\theta. \end{aligned} \tag{4.3.4}$$

Differentiating a second time gives

$$\begin{aligned} \Delta u_1 &= \frac{1}{r}\big(r(u_1)_r\big)_r + \frac{1}{r^2}(u_1)_{\phi\phi} \\ &= \frac{1}{r}\theta_r \cos\phi\cos\theta + \theta_{rr}\cos\phi\cos\theta - \theta_r^2\cos\phi\sin\theta - \frac{1}{r^2}\cos\phi\sin\theta, \\ \Delta u_2 &= \frac{1}{r}\theta_r \sin\phi\cos\theta + \theta_{rr}\sin\phi\cos\theta - \theta_r^2\sin\phi\sin\theta - \frac{1}{r^2}\sin\phi\sin\theta, \\ \Delta u_3 &= \frac{-1}{r}\theta_r\sin\theta - \theta_{rr}\sin\theta - \theta_r^2\cos\theta. \end{aligned} \tag{4.3.5}$$

We choose to consider u_3 now, but remark that we would derive exactly the same formula by considering u_1 or u_2. Substituting (4.3.3) – (4.3.5) into (4.3.2), we obtain

$$\begin{aligned} 0 &= \Delta u_3 + u_3|\nabla u|^2 + \frac{1}{f}\nabla f * \nabla u_3 \\ &= \Big(\frac{-1}{r}\theta_r\sin\theta - \theta_{rr}\sin\theta - \theta_r^2\cos\theta\Big) + \Big(\theta_r^2 + \frac{1}{r^2}\sin^2\theta\Big)\cos\theta - \frac{1}{f}f_r\theta_r\sin\theta \\ &= \frac{-1}{r}\theta_r\sin\theta - \theta_{rr}\sin\theta + \frac{1}{r^2}\sin^2\theta\cos\theta - \frac{1}{f}f_r\theta_r\sin\theta \\ &= \frac{-1}{r^2}\sin\theta\Big(r^2\theta_{rr} + \theta_r\big(r + r^2\frac{1}{f}f_r\big) - \sin\theta\cos\theta\Big). \end{aligned} \tag{4.3.6}$$

□

4.3.2 Example

Define $\theta : [0,1] \to [0,\pi]$ and $f : D \to (0,\infty)$ by

$$\begin{aligned} f(r) &:= \exp\Big(\int_0^r \frac{-\pi s + \cos(\pi s)\sin(\pi s)}{\pi s^2}\, ds\Big), \\ \text{and} \qquad \theta(r) &:= \pi r, \end{aligned} \tag{4.3.7}$$

Then, as we will see in the next section, θ and f satisfy (4.3.1) so $u : (r,\phi) \mapsto (\phi, \theta(r))$ is f-harmonic. We remark that f is smooth and that for small r, $f(r) \approx e^{-\frac{1}{2}\pi^2 r^2}$.

4.3.3 Sequence of f_n-harmonic maps.

We now define a sequence of longitudinally symmetric f-harmonic maps. The reader will notice that $\theta(r,0) = \pi r$ is the $f(\cdot,0)$-harmonic map given in the previous subsection.

For each $n \in \{0,1,2,3,\ldots\}$, define $\theta(\cdot,n) : [0,1] \to [0,\pi]$ by

$$\theta(r,n) := a_n r + \psi(r,n), \tag{4.3.8}$$

where

$$\psi(r,n) := \arccos\left(\frac{1-n^2r^2}{1+n^2r^2}\right),$$

and

$$a_n := \pi - \arccos\left(\frac{1-n^2}{1+n^2}\right) \in (0,\pi].$$

Now define $f(\cdot,n) : D \to (0,\infty)$ by

$$f(r,n) := b_n \exp\Big(\int_0^r F(s,n)\, ds\Big), \tag{4.3.9}$$

where

$$\begin{aligned} F(s,n) := &\frac{4n^3 s}{(1+n^2s^2)(a_n + a_n n^2 s^2 + 2n)} \\ &+ \left(\frac{1+n^2s^2}{a_n + a_n n^2 s^2 + 2n}\right)\left(\frac{\sin(a_n s)\cos\psi\cos\theta - a_n s}{s^2}\right) \\ &+ \left(\frac{2n}{a_n + a_n n^2 s^2 + 2n}\right)\left(\frac{\cos(a_n s)\cos\theta - 1}{s}\right), \end{aligned}$$

and

$$b_n := \exp\Big(-\int_0^1 F(s,n)\, ds\Big).$$

Figure 4.4 shows plots of $\theta(\cdot,n)$, $F(\cdot,n)$ and $f(\cdot,n)$ for a selection of values of n.

Lemma 4.3.2. *For each $n \in \{0, 1, 2, 3, \ldots\}$, the functions $\theta(\cdot, n)$ and $f(\cdot, n)$ satisfy (4.3.1).*

Proof.
Since $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$, we have that

$$\theta_r(r, n) = a_n + \frac{2n}{1 + n^2 r^2}$$

and

$$\theta_{rr}(r, n) = \frac{-4n^3 r}{(1 + n^2 r^2)^2} \ .$$

As $\cos \psi = \frac{1-n^2r^2}{1+n^2r^2}$, we see that $\sin \psi = \frac{2nr}{1+n^2r^2}$. So

$$\frac{\sin\theta\cos\theta - r^2\theta_{rr} - r\theta_r}{r^2\theta_r} =$$

$$\begin{aligned}
&= \frac{\sin\theta\cos\theta + \frac{4n^3r^3}{(1+n^2r^2)^2} - a_n r - \frac{2nr}{1+n^2r^2}}{a_n r^2 + \frac{2nr^2}{1+n^2r^2}} \\
&= (1 + n^2r^2)\frac{\frac{4n^3r^3}{(1+n^2r^2)^2} + \sin(a_n r)\cos\psi\cos\theta - a_n r + \cos(a_n r)\sin\psi\cos\theta - \frac{2nr}{1+n^2r^2}}{a_n(1 + n^2r^2)r^2 + 2nr^2} \\
&= \frac{\frac{4n^3r^3}{(1+n^2r^2)} + (1 + n^2r^2)\left[\sin(a_n r)\cos\psi\cos\theta - a_n r\right] + 2nr\cos(a_n r)\cos\theta - 2nr}{(a_n + a_n n^2 r^2 + 2n)r^2} \\
&= \frac{4n^3 r}{(1 + n^2r^2)(a_n + a_n n^2 r^2 + 2n)} \\
&\quad + \left(\frac{1 + n^2r^2}{a_n + a_n n^2r^2 + 2n}\right)\left(\frac{\sin(a_n r)\cos\psi\cos\theta - a_n r}{r^2}\right) \\
&\quad + \left(\frac{2n}{a_n + a_n n^2r^2 + 2n}\right)\left(\frac{\cos(a_n r)\cos\theta - 1}{r}\right) \\
&= F(r, n) = \frac{f_r(r, n)}{f(r, n)}.
\end{aligned}$$

Therefore (4.3.1) is satisfied, and $(r, \phi) \mapsto (\phi, \theta(r, n))$ is an $f(\cdot, n)$-harmonic map for each $n \in \{0, 1, 2, 3, \ldots\}$. $\square$

Remark. We remark that the sequence $\theta(\cdot, n)$ bubbles as $n \to \infty$. Indeed, we see here that $\theta_r(0, n) = a_n + 2n \to \infty$ as $n \to \infty$.

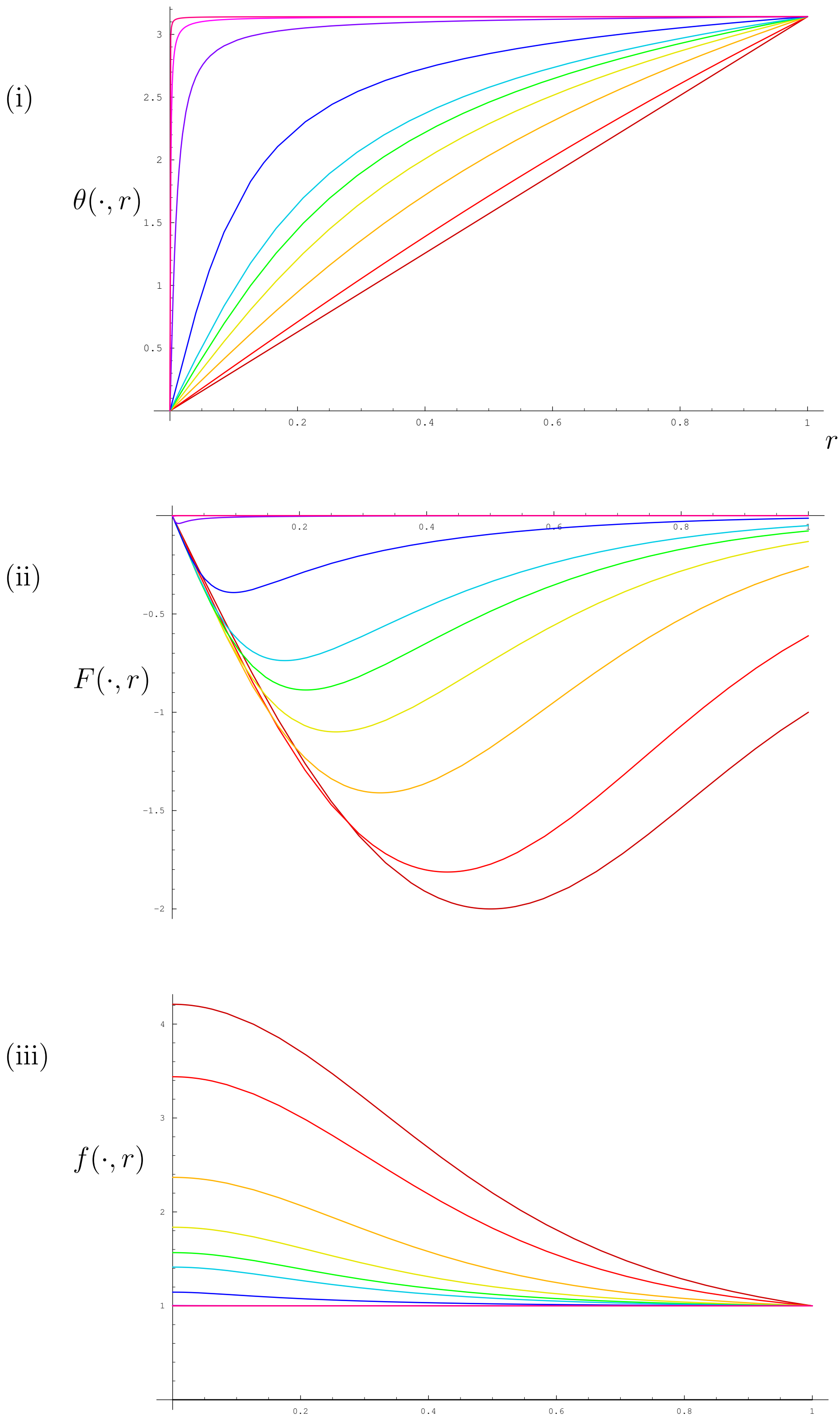

Figure 4.4: The functions (i) $\theta(\cdot, n) : [0, 1] \rightarrow [0, \pi]$, (ii) $F(\cdot, n)$ $(= f_r/f) : [0, 1] \rightarrow [0, \infty)$ and (iii) $f(\cdot, n) : D \rightarrow (0, \infty)$ for the values $n = 0$, 1, 2, 3, 4, 5, 10, 100, 1000 and 10000.

Remark. Notice from figure 4.4 (ii) that, for small r, the derivatives of $F(r, n)$ for the various values of n are almost identical. We will discuss this again in §6.1.

Chapter 5

Bubbles sliding down hills

5.1 Comment

Where do bubbles form? As seen in chapter 3, the f-harmonic map heat flow is smooth away from finitely many singular, or "bubble", points. But can we predict where these bubble points will occur? In the case of infinite time bubbles forming in the interior of $\mathcal{M}$, the answer is yes – they can only exist at critical points of the f.

Intuitively, imagine a bubble trying to form at a non-critical point of f. As the bubble is forming, it will be inclined to "slide" down the "slope" of f so as to try to decrease the f-harmonic energy stored in the bubble. This result is stated precisely in the main theorem of this chapter (Theorem 5.2.1).

The question *"is it possible for an infinite time bubble to form at a non-critical point of f, on the boundary of $\mathcal{M}$,"* is open. As we shall see in the next chapter, it is certainly possible for bubbles forming at finite time to form at non-critical points of f (see §6.2). Going back to our intuitive picture, imagine the bubble simultaneously forming and sliding down f. If the former process is progressing at a fast enough rate, it *may* be possible for the bubble to completely form before it has "slid" all of the way down the slope of f to a critical point.

5.2 The Moving Bubble Theorem

Theorem 5.2.1 (Moving Bubble). *Let $\mathcal{M}$ be a smooth, compact, Riemannian surface, with or without with boundary. Let $z_0 \in \operatorname{int}\mathcal{M}$ and $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$. Moreover if $\partial\mathcal{M}$ is non-empty, suppose further that $u_0|_{\partial\mathcal{M}} \in$*

$C^{2,\alpha}(\partial\mathcal{M};\mathcal{N})$. *Suppose that the map* $u : \mathcal{M} \times [0,\infty) \to \mathcal{N}$ *solves*

$$\begin{cases} u_t - f(x)\Delta_{\mathcal{M}} u = f(x)A(u)(\nabla u, \nabla u) + \nabla f * \nabla u \\ u|_{t=0} = u_0 \\ u(\cdot,t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}. \end{cases} \tag{1.5.1}$$

Suppose further that (z_0,∞) *is a singular (or bubble) point of* u *– as described by Theorem 3.1.1. Then* $\nabla f(z_0) = 0$.

Remark. The proof of this result could be adapted to show that any sequence $\{u_n\} \subset C^\infty(\mathcal{M};\mathcal{N})$ of maps satisfying: (i) $E_f(u_n)$ is bounded; (ii) $u_n \equiv u_1$ on $\partial\mathcal{M}$; and (iii) $\|\tau_f(u_n)\|_{L^2} \to 0$; can only bubble at interior critical points of f. Such a result would also require analogues of the bubbling results of chapter 3.

Theorem 5.2.1 may be restated as:

Corollary 5.2.1.1. *Let* $\mathcal{M}$ *be a smooth, compact, Riemannian surface, possibly with boundary. Suppose that* $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$, $u_0|_{\partial\mathcal{M}} \in C^{2,\alpha}(\partial\mathcal{M};\mathcal{N})$ *and that the map* $u : \mathcal{M}\times[0,\infty) \to \mathcal{N}$ *solves* (1.5.1). *Let* $\Omega \subset \operatorname{int}\mathcal{M}$ *be an open set such that* $\nabla f(x) \neq 0$ *for all* $x \in \overline{\Omega}$. *Then there exists a sequence* $t_n \to \infty$ *such that* $u(\cdot,t_n)|_\Omega$ *converges smoothly.*

5.3 Proof of Theorem 5.2.1

The strategy in the subsequent proof is as follows: Start by assuming that $\nabla f(z_0) \neq 0$. Suppose that the sequence $t_n \to \infty$ is as described in Theorem 3.1.1. We are then able to construct a variation of $u(\cdot,t_n)$ (see figure 5.1 on page 67) so as to "slide" the forming bubble down the slope of f, and show that the (absolute) rate of change of the energy E_f is bounded below. However, this provides a contradiction with the fact that $\|\tau_f(u(\cdot,t_n))\|_{L^2} \to 0$.

Proof of Theorem 5.2.1.

By taking isothermal coordinates in a neighbourhood Ω of z_0, we may assume without loss of generality that Ω is the flat disc $D \subset \mathbb{R}^2$ and that $z_0 = 0$. Suppose, contrary to the proposition, that

$$\nabla f(0) = (-\mu, 0) \tag{5.3.1}$$

for some $\mu > 0$. We may assume (by scaling a finite amount in the domain) that no other bubbles form in $\bar{B}_1 := \bar{B}_1(0)$. We may also assume that $\frac{\partial f}{\partial x_1} < \delta < 0$ on $B_{\frac{1}{2}}$ for some constant δ (where $x = (x_1, x_2)$).

Let $t_n \to \infty$ be the sequence described in Theorem 3.1.1. Define $u_n := u(\cdot, t_n)$. Then $\|\tau_f(u_n)\|_{L^2} \to 0$ as $n \to \infty$ (by choice of (t_n)).

We wish to let K_0 be the "(harmonic) energy lost in the bubble(s) at z_0". Precisely, define

$$K_0 := \lim_{r \searrow 0} \lim_{n \to \infty} E_1\big(u_n; B_r\big) > 0$$

(here E_1 denotes, of course, the *harmonic* energy). Fix $\varepsilon > 0$. By scaling (again) in the domain by a finite amount (independent of n), we claim that we may assume the following: For sufficiently large n,

$$E_f\big(u_n; B_1 \setminus B_{\frac{1}{4}}\big) < \varepsilon \frac{\mu K_0}{32}. \tag{5.3.2}$$

Indeed for $r \in (0,1]$, we know that on $\overline{B}_r \setminus B_{\frac{r}{4}}$, $u_n \to u_\infty$ in C^∞. Since $u_\infty \in C^\infty(\overline{B}_r \setminus B_{\frac{r}{4}}; \mathcal{N})$, we have that $\frac{1}{2}|\nabla u_\infty|^2 < L$ on $B_r \setminus B_{\frac{r}{4}}$ (for some L). Thus

$$\begin{aligned} E_f(u_n; B_r \setminus B_{\frac{r}{4}}) &\to E_f(u_\infty; B_r \setminus B_{\frac{r}{4}}) \\ &< L \|f\|_{L^\infty} \operatorname{Area}(B_r \setminus B_{\frac{r}{4}}) \le Cr^2, \end{aligned}$$

where $C = C(\max(f), u_\infty)$. Therefore by taking $r \in (0,1]$ sufficiently small, we may assume that

$$E_f\big(u_n; B_r \setminus B_{\frac{r}{4}}\big) < \varepsilon \frac{K_0}{32} r \left(-\frac{\partial f}{\partial x_1}(0) \right) \tag{5.3.3}$$

for sufficiently large n. Notice that if we scale (in the domain) by a factor of r, i.e. if $\tilde{f}(x) = f(rx)$ and $\tilde{u}_n(x) = u_n(rx)$, then $(-\tilde{\mu}, 0) := \nabla \tilde{f}(0) = r \nabla f(0)$, so (5.3.3) becomes (5.3.2).

Define a sequence of constants

$$K_n := \frac{1}{2} \int_{B_{\frac{1}{2}}} \Big(-\frac{\partial f}{\partial x_1} \Big) \big|\nabla u_n\big|^2 \, dx.$$

For sufficiently large n, we have that $K_n \ge \frac{1}{2}\mu K_0 > 0$. Now take a test function $\phi \in C^\infty(\mathcal{M}; \mathbb{R})$ with the properties that $\phi|_{B_{\frac{1}{4}}} \equiv 1$ and $\operatorname{supp}(\phi) \subset B_{\frac{1}{2}}$. We may assume that $|\nabla \phi| < 8$. Define a family of diffeomorphisms $\eta^{(n)}_{\cdot} : \mathcal{M} \times (-\xi, \xi) \to \mathcal{M}$ by $\eta^{(n)}_s(x) = x - s(\frac{\phi}{K_n}, 0)$ (see figure 5.1). Notice that $\eta_s(x) \equiv x$ on $\mathcal{M} \setminus B_1(0)$ for all s. We drop the "$\,^{(n)}$" notation on η in the name of brevity. We remark that it is important here that K_n is bounded below, so that the variations have bounded "velocity" (i.e. $\frac{\partial \eta}{\partial s}$ is bounded).

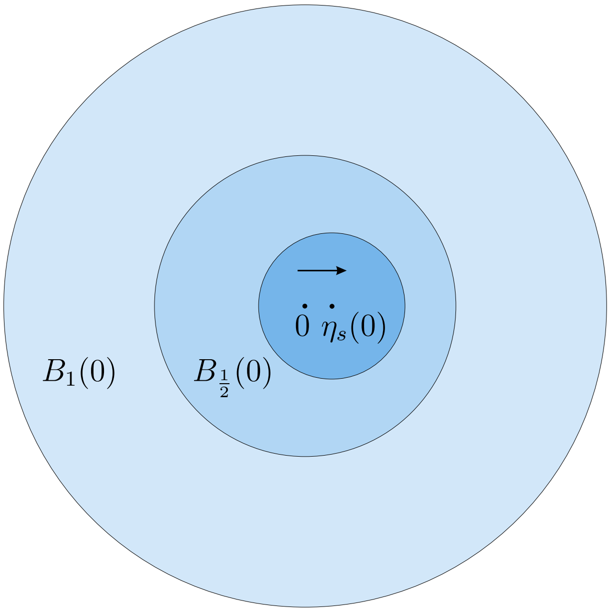

Figure 5.1: Using a deformation of the domain D to move the bubble(s) "downhill", or down the slope of f.

So,

$$
\begin{aligned}
E_f\big(u_n \circ \eta_s; \mathcal{M}\big) &= \frac{1}{2}\int_{B_1} f(x)\left|\nabla\big(u_n \circ \eta_s\big)(x)\right|^2\, dx + E_f(u_n; \mathcal{M}\setminus B_1) \\
&= \frac{1}{2}\int_{B_1} f(x)\sum_{j=1}^{2}\left|\sum_{i=1}^{2}\frac{\partial u_n}{\partial x_i}\big(\eta_s(x)\big)\frac{\partial \eta_s^i}{\partial x_j}(x)\right|^2\, dx + E_f(u_n; \mathcal{M}\setminus B_1) \\
&= \frac{1}{2}\int_{B_1} f(x)\Bigg[\left|\frac{\partial u_n}{\partial x_1}\big(\eta_s(x)\big)\right|^2\left(1 - \frac{s}{K_n}\frac{\partial \phi}{\partial x_1}(x)\right)^2 \\
&\qquad + \left|\frac{\partial u_n}{\partial x_1}\big(\eta_s(x)\big)\right|^2\left(\frac{s}{K_n}\frac{\partial \phi}{\partial x_2}(x)\right)^2 + \left|\frac{\partial u_n}{\partial x_2}\big(\eta_s(x)\big)\right|^2 \\
&\qquad - \left\langle \frac{\partial u_n}{\partial x_1}\big(\eta_s(x)\big), \frac{\partial u_n}{\partial x_2}\big(\eta_s(x)\big)\right\rangle\left(\frac{2s}{K_n}\frac{\partial \phi}{\partial x_2}(x)\right)\Bigg]\, dx \\
&\qquad + E_f(u_n; \mathcal{M}\setminus B_1)
\end{aligned}
$$

Differentiating, and then integrating by parts gives us

$$
\begin{aligned}
&\frac{d}{ds}E_f\big(u_n \circ \eta_s; \mathcal{M}\big)\Big|_{s=0} \\
&\quad = \frac{1}{2}\int_{B_1} f(x)\left(\frac{\partial}{\partial x_1}\left|\frac{\partial u_n}{\partial x_1}(x)\right|^2\right)\left(\frac{-\phi}{K_n}\right)\,dx \\
&\qquad + \frac{1}{2}\int_{B_1} f(x)\left|\frac{\partial u_n}{\partial x_1}(x)\right|^2\left(-\frac{2}{K_n}\frac{\partial \phi}{\partial x_1}(x)\right)\,dx \\
&\qquad + \frac{1}{2}\int_{B_1} f(x)\frac{\partial}{\partial x_1}\left|\frac{\partial u_n}{\partial x_2}(x)\right|^2\left(\frac{-\phi}{K_n}\right)\,dx \\
&\qquad - \frac{1}{2}\int_{B_1} f(x)\left\langle\frac{\partial u_n}{\partial x_1}(x), \frac{\partial u_n}{\partial x_2}(x)\right\rangle\left(\frac{2}{K_n}\frac{\partial \phi}{\partial x_2}(x)\right)\,dx \\
&\quad = \frac{1}{2K_n}\int_{B_{\frac{1}{2}}}\frac{\partial f}{\partial x_1}\phi\left|\nabla u_n\right|^2\,dx + \frac{1}{2K_n}\int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} f\left|\nabla u_n\right|^2\frac{\partial \phi}{\partial x_1}\,dx \\
&\qquad - \frac{1}{K_n}\int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} f\left|\frac{\partial u_n}{\partial x_1}\right|^2\frac{\partial \phi}{\partial x_1}\,dx - \frac{1}{K_n}\int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} f\left\langle\frac{\partial u_n}{\partial x_1}, \frac{\partial u_n}{\partial x_2}\right\rangle\frac{\partial \phi}{\partial x_2}\,dx.
\end{aligned}
$$

Now from (5.3.2), we may deduce that for sufficiently large n

$$
\frac{3}{4}K_n \le -\frac{1}{2}\int_{B_{\frac{1}{2}}}\frac{\partial f}{\partial x_1}\phi\left|\nabla u_n\right|^2\,d\underline{x} \le \frac{1}{2}\int_{B_{\frac{1}{2}}}\left(-\frac{\partial f}{\partial x_1}\right)\left|\nabla u_n\right|^2\,d\underline{x} = K_n
$$

with the first inequality holding if ε is sufficiently small. So, by (5.3.2), $2K_n \ge \mu K_0$ and $|\nabla\phi| < 8$, we see that

$$
\begin{aligned}
&\left|\frac{d}{ds}E_f\big(u_n \circ \eta_s; \mathcal{M}\big)\Big|_{s=0} + 1\right| \\
&\qquad \le \frac{1}{4} + \left|\frac{1}{K_n}\int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} f\left|\frac{\partial u_n}{\partial x_1}\right|^2\frac{\partial \phi}{\partial x_1}\,d\underline{x}\right| + \left|\frac{1}{2K_n}\int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} f\left|\nabla u_n\right|^2\frac{\partial \phi}{\partial x_1}\,d\underline{x}\right| \\
&\qquad\quad + \left|\frac{1}{2K_n}\int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} f\left|\nabla u_n\right|^2\frac{\partial \phi}{\partial x_2}\,d\underline{x}\right| \\
&\qquad \le \frac{1}{4} + \frac{32}{\mu K_0}\int_{B_{\frac{1}{2}}\setminus B_{\frac{1}{4}}} f\left|\nabla u_n\right|^2\,d\underline{x} < \frac{1}{4} + \varepsilon.
\end{aligned}
\tag{5.3.4}
$$

But this means that for sufficiently large n,

$$\left| \frac{d}{ds} E_f\big(u_n \circ \eta_s; \mathcal{M}\big)\Big|_{s=0} \right| > \frac{1}{2}. \tag{5.3.5}$$

We argue that this leads to a contradiction. Indeed, because $\|\tau_f(u_n)\|_{L^2} \to 0$ and because

$$\begin{aligned} \left\| \frac{d}{ds}\left(u_n \circ \eta_s\right)\Big|_{s=0} \right\|^2_{L^2(\mathcal{M})} &= \frac{1}{K_n^2} \int_{\mathcal{M}} \phi^2 \left| \frac{\partial u_n}{\partial x_1} \right|^2 d\mathcal{M} \\ &\leq \frac{1}{K_n^2} \int_{\mathcal{M}} \left| \nabla u_n \right|^2 d\mathcal{M} \\ &\leq \frac{c(f)}{K_0^2} E_f(u_n) < L, \end{aligned} \tag{5.3.6}$$

for some constant $L < \infty$ independent of n, we have that

$$\begin{aligned} \frac{d}{ds} E_f\left(u_n \circ \eta_s\right)\Big|_{s=0} &= \frac{1}{2} \frac{d}{ds} \int_{\mathcal{M}} f \left| \nabla\left(u_n \circ \eta_s\right) \right|^2 d\mathcal{M} \Big|_{s=0} \\ &= \frac{1}{2} \int_{\mathcal{M}} f \left\langle \nabla \frac{d}{ds}\left(u_n \circ \eta_s\right), \nabla u_n \right\rangle d\mathcal{M} \Big|_{s=0} \\ &= -\frac{1}{2} \int_{\mathcal{M}} \left\langle \frac{d}{ds}\left(u_n \circ \eta_s\right)\Big|_{s=0}, \tau_f(u_n) \right\rangle d\mathcal{M} \\ &\to 0. \end{aligned} \tag{5.3.7}$$

This contradicts (5.3.5). Therefore we must have that $\nabla f(z_0) = 0$. □

Chapter 6

Comments on the sharpness of the Moving Bubble Theorem

In this section, we discuss the sharpness of the "Moving Bubble Theorem"[1].

6.1 An infinite time bubble forming at the maximum of f.

6.1.1 Statement

Recall the f-harmonic heat flow equation;

$$\begin{cases} u_t - f(x)\Delta_{\mathcal{M}} u = f(x)A(u)(\nabla u, \nabla u) + \nabla f * \nabla u \\ u|_{t=0} = u_0 \\ u(\cdot, t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}. \end{cases} \tag{1.5.1}$$

We have seen that bubbles forming at infinite time, must form at a critical point of the weighting function f. A hypothesis might be made that, in order to minimize energy, such bubbles would only form at minima of f. This would be false. In this section, we give an example of an f-harmonic heat flow where a bubble forms at the *maximum* of the f at infinite time.

As in §4.3 we work with longitudinally symmetric maps. We check first that the heat flow applied to a longitudinally symmetric map, always yields a longitudinally symmetric flow. Recall the definition of U_α on page 58.

Lemma 6.1.1. *Suppose that $f : D \to (0, \infty)$ is rotationally symmetric. Suppose that $u : D \times [0, \infty) \to S^2$ solves the heat flow equation (1.5.1).*

[1] Theorem 5.2.1.

If $u_0 = U_{\theta_0}$ is longitudinally symmetric, then so is $u(\cdot, t)$ for $t \in (0, T)$. Moreover, if $u(\cdot, t) = U_{\theta(|\cdot|, t)}$, then θ solves

$$\begin{cases} \frac{1}{f}\theta_t = \theta_{rr} + \frac{1}{r}\theta_r + \frac{f_r}{f}\theta_r - \frac{1}{r^2}\sin\theta\cos\theta \\ \theta|_{t=0} = \pi r. \end{cases} \tag{6.1.1}$$

Remark. The proof of this lemma is omitted. See Chang-Ding [CD90, Lemma 2.2] for the equivalent result for harmonic maps. Their proof adapts straightforwardly to the f-harmonic case.

By placing a restriction on how large the slope of f may be, we see that it is possible for an infinite time bubble to form at the maximum of f.

Lemma 6.1.2. *Suppose that $f : D \to (0, \infty)$ is rotationally symmetric, and satisfies*

$$-\frac{\pi^4 r^3}{576} \le \frac{f_r}{f} \le 0. \tag{6.1.2}$$

Suppose that $u : D \times [0, \infty) \to S^2$ solves the heat flow equation (1.5.1), where $u_0(r, \phi) = (\phi, \pi r)$ in polar coordinates. Then a bubble forms at the origin – the maximum of f – at infinite time.

Remark. The bound on f_r/f is not sharp.

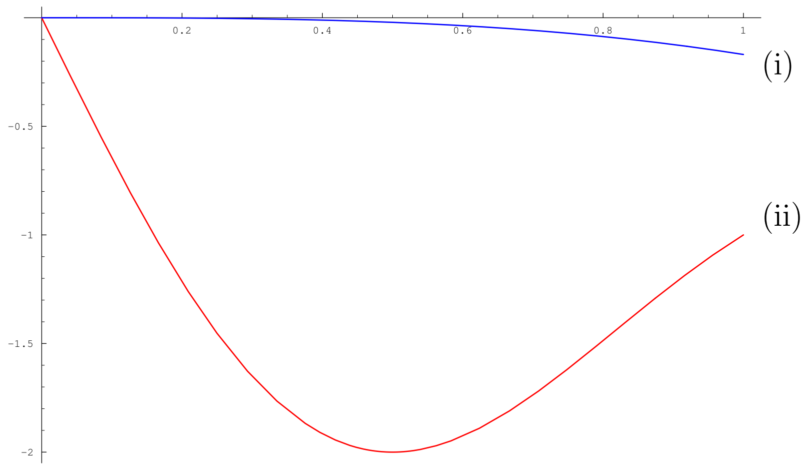

Figure 6.1: Plots of (i) $-\frac{\pi^4 r^3}{576}$ and (ii) $\frac{-\pi s + \cos(\pi s)\sin(\pi s)}{\pi s^2}$.

Remark. Recall the example given in §4.3 of $f(r) = e^{\int_0^r \frac{-\pi s + \cos(\pi s)\sin(\pi s)}{\pi s^2}\, ds}$. This f is sufficiently larger at the origin, than at the boundary, so prevents bubbling at that point (given suitable initial condition such as $\theta(r,0) = \pi r$) and the symmetry prevents a bubble forming anywhere else. Recall that this f is approximately equal to e^{-cr^2} (some constant c) near the origin. Compare that with a function such as $\bar{f}(r) = e^{-\frac{\pi^4 r^4}{2304}}$ (i.e. such that $\bar{f}_r/\bar{f} = \frac{-\pi^4 r^3}{576}$). This function $\bar{f}$ is not significantly larger at the origin than at the boundary and, as Lemma 6.1.2 shows, is unable to prevent the considered flow from bubbling. Figure 6.1 shows plots of f_r/f and $\bar{f}_r/\bar{f}$.

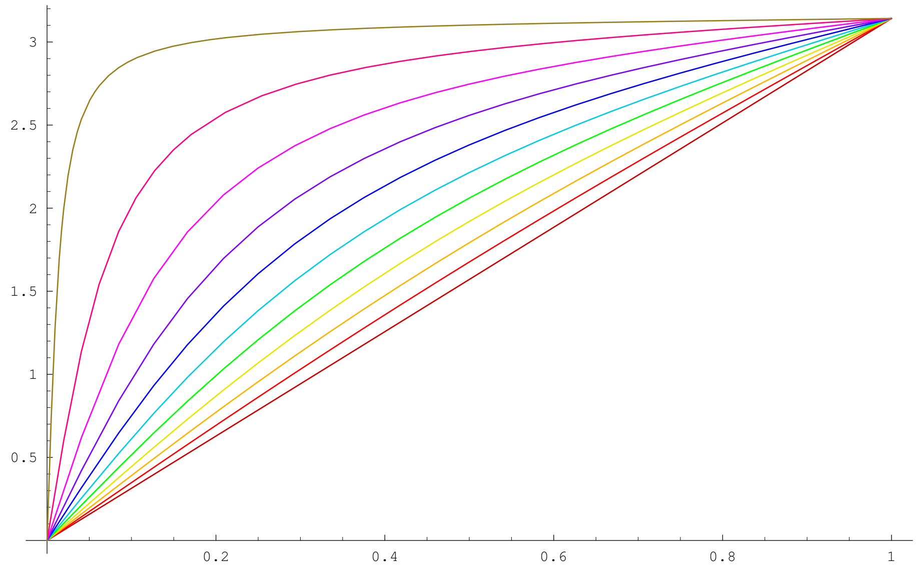

Figure 6.2: Plots of $\alpha(\cdot,\varepsilon)$ for the values $\varepsilon = 0.5, 0.45, 0.4, 0.35, 0.3, 0.25, 0.2, 0.15, 0.1, 0.05$ and 0.01.

6.1.2 Proof of Lemma 6.1.2

Proof of Lemma 6.1.2.

By constructing a sub-solution to $\theta(r,t)$, we will show that the solution to (6.1.1) must form a bubble at the origin. We may suppose that $\min(f) = 1$. Consider the family of maps $\alpha(\cdot,t) : [0,1] \to [0,\pi]$, defined by

$$\alpha(r,\varepsilon) := 2\arctan\left[\varepsilon\left(\tan\left(\frac{\pi r}{4}\right) - \cot\left(\frac{\pi r}{4}\right)\right)\right] + \pi \qquad (6.1.3)$$

where $\varepsilon \in [0,1]$ (c.f. [Top04a, §3]). See figure 6.2. Notice that $\alpha(0,\varepsilon) = 0$ and $\alpha(1,\varepsilon) = \pi$. Notice further that $\alpha(r,\frac{1}{2}) = \pi r$. Now suppose that ε :

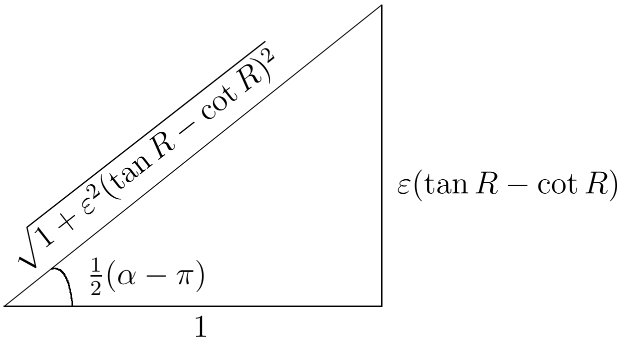

Figure 6.3: $\alpha = 2\arctan\left[\varepsilon\left(\tan R - \cot R\right)\right] + \pi$.

$[0, \infty) \to [0, \frac{1}{2}]$ solves the ODE

$$\begin{cases} \varepsilon_t(t) &= -\pi^2 \varepsilon^3(t)\ , \\ \varepsilon(0) &= \frac{1}{2}, \end{cases} \tag{6.1.4}$$

and consider $\alpha(r, \varepsilon(t))$. We show that $\alpha(r, \varepsilon(t))$ satisfies

$$\begin{cases} \frac{1}{f}\alpha_t \leq \alpha_{rr} + \frac{1}{r}\alpha_r + \frac{f_r}{f}\alpha_r - \frac{1}{r^2}\sin\alpha\cos\alpha, \\ \alpha(r, \varepsilon(0)) = \pi r. \end{cases} \tag{6.1.5}$$

There follows a lengthy calculation to verify this differential equation. This calculation is not especially interesting and the reader should feel free to skip large sections of it, comfortable in the knowledge that it has been verified with the program Mathematica ®.

We write $R := \frac{\pi r}{4}$. Since $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$, differentiating twice gives

$$\begin{aligned} \alpha_r &= \frac{\frac{1}{2}\varepsilon\pi(\sec^2 R + \operatorname{cosec}^2 R)}{1+\varepsilon^2(\tan R - \cot R)^2} = \frac{\frac{1}{2}\varepsilon\pi}{(1+\varepsilon^2(\frac{\sin R}{\cos R} - \frac{\cos R}{\sin R})^2)\sin^2 R\cos^2 R} \\ &= \frac{\frac{1}{2}\varepsilon\pi}{\sin^2 R\cos^2 R + \varepsilon^2(\sin^2 R - \cos^2 R)^2} \\ &= \frac{4\varepsilon\pi}{2\sin^2 2R + 8\varepsilon^2\cos^2 2R - 4\varepsilon^2\sin^2 2R + 4\varepsilon^2\sin^2 2R + 1 - \sin^2 2R - \cos^2 2R} \\ &= \frac{4\varepsilon\pi}{1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r}, \end{aligned}$$

and

$$\alpha_{rr} = \frac{4\varepsilon(-1+4\varepsilon^2)\pi^2\sin\pi r}{(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r)^2}.$$

From figure 6.3 on page 73, we see that

$$\sin\frac{\alpha-\pi}{2} = \frac{\varepsilon(\tan R - \cot R)}{\sqrt{1+\varepsilon^2(\tan R-\cot R)^2}}\,, \qquad \cos\frac{\alpha-\pi}{2} = \frac{1}{\sqrt{1+\varepsilon^2(\tan R-\cot R)^2}}\,.$$

So

$$\begin{aligned}
&\sin\alpha\cos\alpha \\
&\quad= \big(-\sin(\alpha-\pi)\big)\big(-\cos(\alpha-\pi)\big) \\
&\quad= 2\sin\frac{1}{2}(\alpha-\pi)\cos\frac{1}{2}(\alpha-\pi)\Big(\cos^2\frac{1}{2}(\alpha-\pi)-\sin^2\frac{1}{2}(\alpha-\pi)\Big) \\
&\quad= \frac{2\varepsilon(\tan R-\cot R)\big(1-\varepsilon^2(\tan R-\cot R)^2\big)}{\big(1+\varepsilon^2(\tan R-\cot R)^2\big)^2} \\
&\quad= \frac{2\varepsilon(\frac{\sin^2 R-\cos^2 R}{\sin R\cos R})\big(1-\varepsilon^2(\frac{\sin^2 R-\cos^2 R}{\sin R\cos R})^2\big)}{\big(1+\varepsilon^2(\frac{\sin^2 R-\cos^2 R}{\sin R\cos R})^2\big)^2} \\
&\quad= \frac{128\varepsilon\sin R\cos R(\sin^2 R-\cos^2 R)\big(\sin^2 R\cos^2 R-\varepsilon^2(\sin^2 R-\cos^2 R)^2\big)}{\big(8\sin^2 R\cos^2 R+8\varepsilon^2(\sin^2 R-\cos^2 R)^2\big)^2} \\
&\quad= \frac{8\varepsilon\sin 2R(-\cos 2R)\big(2\sin^2 2R-8\varepsilon^2\cos^2 2R\big)}{\big(2\sin^2 2R+8\varepsilon^2\cos^2 2R\big)^2} \\
&\quad= \frac{-4\varepsilon\sin\pi r\big(1-4\varepsilon^2+(-1-4\varepsilon^2)\cos\pi r\big)}{\big(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r\big)^2}\,.
\end{aligned}$$

Furthermore

$$\frac{\partial\alpha}{\partial t}\big(\theta,\varepsilon(t)\big) = \frac{2\varepsilon_t(\tan R-\cot R)}{1+\varepsilon^2(\tan R-\cot R)^2} = \frac{-4\varepsilon_t\sin\pi r}{1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r}\,.$$

Therefore

$$
\begin{aligned}
Q &:= \left[\alpha_{rr} + \frac{1}{r}\alpha_r + \frac{f_r}{f}\alpha_r - \frac{1}{r^2}\sin\alpha\cos\alpha\right] - \frac{1}{f}\alpha_t \\
&= \frac{4\varepsilon(-1+4\varepsilon^2)\pi^2\sin\pi r}{(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r)^2} + \left(\frac{1}{r}+\frac{f_r}{f}\right)\frac{4\varepsilon\pi}{1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r} \\
&\quad + \left(\frac{1}{r^2}\right)\frac{4\varepsilon\sin\pi r\big(1-4\varepsilon^2+(-1-4\varepsilon^2)\cos\pi r\big)}{\big(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r\big)^2} \\
&\quad + \left(\frac{1}{f}\right)\frac{4\varepsilon_t\sin\pi r}{1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r} \\
&= \frac{4}{r^2\big(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r\big)^2}\Big[(-1+4\varepsilon^2)\varepsilon\pi^2r^2\sin\pi r \\
&\quad + \left(\pi\varepsilon r + \frac{f_r}{f}\varepsilon\pi r^2 + \frac{\varepsilon_t r^2}{f}\sin\pi r\right)\big(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r\big) \\
&\quad + \varepsilon\sin\pi r\big(1-4\varepsilon^2+(-1-4\varepsilon^2)\cos\pi r\big)\Big] \\
&= \frac{4\varepsilon Q_1 + 16\varepsilon^3 Q_2(1+\cos\pi r) + 4Q_3 r^2\sin\pi r}{r^2\big(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r\big)^2},
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1 &:= \left(\pi r + \frac{f_r}{f}\pi r^2 + \sin\pi r\right)(1-\cos\pi r) - \pi^2 r^2\sin\pi r, \\
Q_2 &:= \pi r + \frac{f_r}{f}\pi r^2 - \sin\pi r, \\
Q_3 &:= 4\varepsilon^3\pi^2 + \frac{\varepsilon_t}{f}\big(1+4\varepsilon^2+(-1+4\varepsilon^2)\cos\pi r\big).
\end{aligned}
$$

Now, as

$$
\begin{aligned}
\sin\pi r &\geq \pi r - \frac{\pi^3 r^3}{3!}, \\
-\cos\frac{\pi r}{2} &\geq -1 + \frac{\pi^2 r^2}{2^2 2!} - \frac{\pi^4 r^4}{2^4 4!}, \\
\sin\frac{\pi r}{2} &\geq \frac{\pi r}{2} - \frac{\pi^3 r^3}{2^3 3!},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
Q_1 &= (\pi r + \frac{f_r}{f}\pi r^2 + \sin \pi r) 2 \sin^2 \frac{\pi r}{2} - 2\pi^2 r^2 \sin \frac{\pi r}{2} \cos \frac{\pi r}{2} \\
&\geq 2 \sin \frac{\pi r}{2} \Bigg[\left(2\pi r + \frac{f_r}{f}\pi r^2 - \frac{\pi^3 r^3}{6} \right) \left(\frac{\pi r}{2} - \frac{\pi^3 r^3}{48} \right) \\
&\quad + \pi^2 r^2 \left(-1 + \frac{\pi^2 r^2}{8} - \frac{\pi^4 r^4}{384} \right) \Bigg] \\
&= 2\pi^2 r^2 \sin \frac{\pi r}{2} \Bigg[1 - \frac{\pi^2 r^2}{24} - \frac{\pi^2 r^2}{12} + \frac{\pi^4 r^4}{288} + \frac{f_r}{f} r \left(\frac{1}{2} - \frac{\pi^2 r^2}{48} \right) \\
&\quad - 1 + \frac{\pi^2 r^2}{8} - \frac{\pi^4 r^4}{384} \Bigg] \\
&= \pi^2 r^2 \sin \frac{\pi r}{2} \left[\frac{\pi^4 r^4}{576} + \frac{f_r}{f} r \left(1 - \frac{\pi^2 r^2}{24} \right) \right] \\
&\geq \pi^2 r^2 \sin \frac{\pi r}{2} \left[\frac{\pi^4 r^4}{576} - \frac{\pi^4 r^4}{576} \left(1 - \frac{\pi^2 r^2}{24} \right) \right] \geq 0.
\end{aligned}
\tag{6.1.6}
$$

Moreover

$$
Q_2 = \pi r - \sin \pi r + \frac{f_r}{f}\pi r^2 \geq \frac{\pi^3 r^3}{3!} - \frac{\pi^5 r^5}{5!} - \frac{\pi^5 r^5}{576} \geq \frac{\pi^3 r^3}{3!} \left(1 - \frac{1}{2} - \frac{10}{96} \right) \geq 0, \tag{6.1.7}
$$

and by (6.1.4) we have that

$$
Q_3 = 4\varepsilon^3 \pi^2 - \frac{\varepsilon^3 \pi^2}{f} \big(1 + 4\varepsilon^2 + (-1 + 4\varepsilon^2) \cos \pi r \big) \geq 2\varepsilon^3 \pi^2 \geq 0 \tag{6.1.8}
$$

since $\min(f) = 1$. It follows from (6.1.6) – (6.1.8) that $Q \geq 0$. Equivalently, we see that $\alpha\big(r, \varepsilon(t)\big)$ satisfies

$$
\begin{cases}
\frac{1}{f}\alpha_t \leq \alpha_{rr} + \frac{1}{r}\alpha_r + \frac{f_r}{f}\alpha_r - \frac{1}{r^2} \sin \alpha \cos \alpha, \\
\alpha(r, \varepsilon(0)) = \pi r.
\end{cases}
\tag{6.1.9}
$$

So by the comparison principle (see [CD90] for details in a similar situation), it follows that θ – the solution to the heat flow (6.1.1) – satisfies

$$
\theta(r, t) \geq \alpha(r, \varepsilon(t))
$$

for all $r \in [0,1]$ and $t \in [0,\infty)$. But as

$$\alpha_r = \frac{4\varepsilon\pi}{1 + 4\varepsilon^2 + (-1+4\varepsilon^2)\cos\pi r}, \tag{6.1.10}$$

we see that

$$\alpha_r(0,\varepsilon(t)) = \frac{\pi}{2\varepsilon(t)} \to \infty \tag{6.1.11}$$

as $t \to \infty$. Therefore, the flow $\theta(r,t)$ must bubble at the origin, in either finite or infinite time. Finally we remark that the 1-harmonic heat flow is a super-solution to θ, so we may rule out bubbling within finite time. $\square$

6.2 A finite time bubble not at a critical point

Recall Theorem 5.2.1 which stated that if an infinite time bubble formed at the point $z_0 \in \operatorname{int}\mathcal{M}$, then z_0 is a critical point of f. Whether or not an infinite time bubble can form at $z_0 \in \partial\mathcal{M}$ if $\nabla f \neq 0$ is unknown. However:

Lemma 6.2.1. *There exists an f-harmonic heat flow, under which a finite time singularity can form at a non-critical point of f.*

First we quote a result of P. Topping; [Top04b, Lemma 1.3]

Lemma 6.2.2 (Topping). *There exist a compact target manifold $\mathcal{N}$, a smooth map $v_0 : D \to \mathcal{N}$ and $\varepsilon > 0$ such that every smooth map $v : D \to \mathcal{N}$ homotopic to v_0 fails to be harmonic, and if $E_1(v) \leq E_1(v_0)$ also, then*

$$\int_D |\tau_1(v)|^2 \geq \varepsilon.$$

Proof of Lemma 6.2.1.

We take $\mathcal{N}$, ε and v_0 as described by Lemma 6.2.2. We may suppose that

$$\varepsilon < \min\left\{\frac{1}{64}, \frac{1}{2}E_1(v_0)^{\frac{1}{2}}\right\}. \tag{6.2.1}$$

Define $f : D \to (0,\infty)$ by $f(x,y) = 1 + \frac{\alpha\varepsilon}{E_1(v_0)^{\frac{1}{2}}}(1-x)$ where $\alpha \in (0,1)$ is a constant to be chosen later. Then $f \geq 1$ and $|\nabla f| = \frac{\alpha\varepsilon}{E_1(v_0)^{\frac{1}{2}}}$. So, if $v : D \to \mathcal{N}$ is homotopic to v_0 and $E_1(v) \leq E_1(v_0)$ then

$$
\begin{aligned}
\int_D |\tau_f(v)|^2 &= \int_D |f\tau_1(v) + \nabla f * \nabla v|^2 \\
&= \int_D f^2|\tau_1(v)|^2 + 2\int_D \langle f\tau(v), \nabla f * \nabla v\rangle + \int_D |\nabla f * \nabla v|^2 \\
&\geq \int_D |\tau_1(v)|^2 - 2\int_D f|\tau(v)||\nabla f||\nabla v| + 0 \\
&\geq \frac{1}{2}\int_D |\tau_1(v)|^2 - 2\int_D f^2|\nabla f|^2|\nabla v|^2 \\
&\geq \frac{1}{2}\int_D |\tau_1(v)|^2 - 4\left[1 + \frac{2\alpha\varepsilon}{E_1(v_0)^{\frac{1}{2}}}\right]^2 \alpha^2\varepsilon^2 \frac{E_1(v)}{E_1(v_0)} \\
&\geq \frac{1}{2}\varepsilon - 16\varepsilon^2 \geq \frac{1}{4}\varepsilon,
\end{aligned}
\tag{6.2.2}
$$

by Lemma 6.2.2 and (6.2.1).

Now, for sufficiently small α, we may choose an initial map u_0 homotopic to v_0, such that $E_f(u_0) \leq E_1(v_0)$. Such a u_0 must exist by the following argument: The harmonic heat flow, v, with initial map v_0, is continuous on some interval $[0, t_1]$. Write $u_0 := v(t_1)$. As $\int_D |\tau(v(t))|^2 \geq \varepsilon$, $E_1(u_0) + \varepsilon t_1 \leq E_1(v_0)$. By choosing α sufficiently small, we see that

$$
E_f(u_0)) \leq \left[1 + \frac{2\alpha\varepsilon}{E_1(v_0)^{\frac{1}{2}}}\right] E_1(u_0) \leq E_1(v_0).
$$

Therefore, as $f \geq 1$, any map u with $E_f(u) \leq E_f(u_0)$ also has $E_1(u) \leq E_f(u) \leq E_f(u_0) \leq E_1(v_0)$. But (6.2.2), together with

$$
\frac{d}{dt}E_f(u(t)) = -\int_D |\tau_f(u(t))|^2 \tag{6.2.3}
$$

and the fact that $E_f \geq 0$, tells us that the (f-)heat flow, with initial map u_0, can certainly not be smooth on the time interval $[0, T]$, if $T > \frac{4E_f(u_0)}{\varepsilon}$. Therefore, there must form a singularity at some point $z_0 \in \overline{D}$ in finite time, but of course $\nabla f(z_0) \neq 0$.

□

Remark. An example (see [Top04b, page 23]) of such an $\mathcal{N}$ is the warped product $S^1 \times_q S^2$ where $q : S^1 \to [1, 3]$ is given by $q(\theta) = 2 - \cos\theta$.

Chapter 7

A suggested application of the Moving Bubble Theorem

7.1 comment

In this final chapter, we provide some motivation for the topics which have been presented in the previous six chapters. We will discuss a "possible" application of the theory presented above, but all arguments given here are deliberately vague – giving precise details would be beyond the scope of this thesis.

We will be considering a heat flow $u : D \times [0, \infty) \to S^2$ with the boundary ∂D mapping to a point. By using a very particular function $f : D \to (0, \infty)$ that we describe below, we hope to find that the set of singular points – that is the set of points $\bar{z} \in D$ with the following property: *On every neighbourhood of $\bar{z}$, the map u does not have uniformly (with t) bounded derivative* – is a line (namely a one dimensional sub-manifold). We might otherwise have guessed (from Theorem 3.1.1) that the set of singular points would be a finite set. We can then take the warped product (see section 1.2) of D with, say, S^1 to find a (1-)harmonic map heat flow from a three dimensional target manifold for which the set of singular points would be a two dimensional sub-manifold – not one-dimensional as might be expected.

7.2 The groove

Let us now describe the function f. Intuitively we choose f so as to make the disc D "look like" a vinyl LP. That is, so that the graph of f contains a

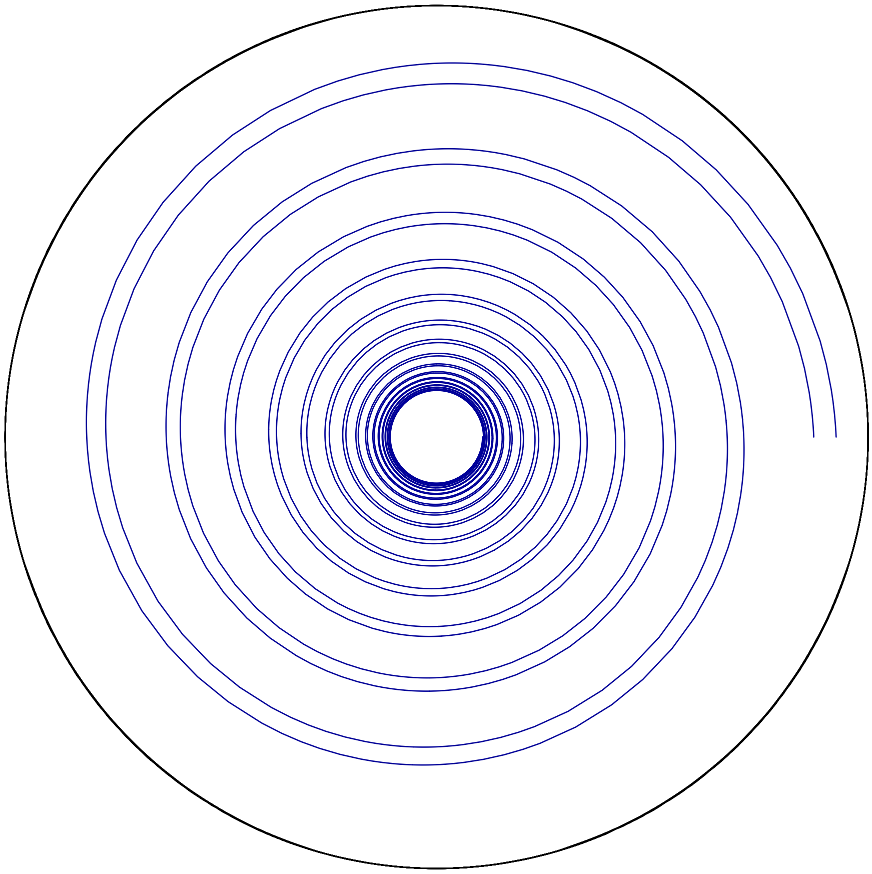

Figure 7.1: The "record groove" between, the plots of $\mu_\pm : [0,\infty) \to D$ given by $\mu_\pm(s) = \big[0.1 + 0.8e^{-\alpha s} \pm 0.1e^{-\alpha s}(1-e^{-\alpha})\big](\cos 2\pi s, \sin 2\pi s)$ for $\alpha = 0.3$.

"spiral groove" – see figure 7.1. Notice that in figure 7.1

$$\begin{aligned}\big|\mu_-(s)\big| - \big|\mu_+(s+1)\big| &= 0.8e^{-\alpha s} - 0.1e^{-\alpha s}(1-e^{-\alpha}) \\ &\qquad - 0.8e^{-\alpha(s+1)} - 0.1e^{-\alpha(s+1)}(1-e^{-\alpha}) \\ &= (1-e^{-\alpha})\left[0.8e^{-\alpha s} - 0.1e^{-\alpha s} - 0.1e^{-\alpha(s+1)}\right] \\ &> 0\end{aligned} \tag{7.2.1}$$

so the spiral does not intersect with itself.

One way of defining such an f is as follows: Let $\alpha > 0$ be some small number. For $z = (r\cos\theta, r\sin\theta) \in D$, define

$$Q(z,\alpha,n) := \left| r - \frac{1}{10} - \frac{4}{5}e^{\frac{-\alpha(\theta+2\pi n)}{2\pi}} \right| - e^{\frac{-\alpha(\theta+2\pi n)}{2\pi}} \left(\frac{1-e^{-\alpha}}{10}\right) \tag{7.2.2}$$

We then define the *groove* on D by

$$G_\alpha := \Big\{ z = (r\cos\theta, r\sin\theta) \in D : \exists n \in \mathbb{N} \cup \{0\} \text{ such that } Q(z,\alpha,n) < 0 \Big\}.$$

Remark. The "width" of the groove (width $= 2e^{\frac{-\alpha(\theta+2\pi n)}{2\pi}} \left(\frac{1-e^{-\alpha}}{10}\right)$) is chosen so as to ensure that the n in the definition of G_α is unique.

For $z \in G_\alpha$, define $s(z) = n + \frac{\theta}{2\pi}$ for the unique $n = n(z)$ which satisfies $Q(z,\alpha,n) < 0$. We can then define f on G_α by

$$f(z) := 1 + F_{\text{groove}}(z) + F_{\text{slope}}(z) \tag{7.2.3}$$

where

$$F_{\text{groove}}(z) := \beta \left| \frac{r - \frac{1}{10} - \frac{4}{5}e^{-\alpha s}}{e^{-\alpha s}\left(\frac{1-e^{-\alpha}}{10}\right)} \right|^2 \tag{7.2.4}$$

and

$$F_{\text{slope}}(z) := \gamma e^{-\alpha s} \tag{7.2.5}$$

for some small $\beta, \gamma > 0$. Extend f to the rest of D in some arbitrary smooth way.

7.3 Sliding down the groove

Let

$$r_0 << 0.1e^{-\alpha}\left(1 - e^{-\alpha}\right), \qquad z_0 = (0.1 + 0.8e^{-\alpha}, 0). \tag{7.3.1}$$

Suppose now that we have some initial map $u_0 : D \to S^2$, which maps ∂D onto the "south pole" 0, maps $D \setminus B_{r_0}(z_0)$ onto a small neighbourhood of 0 and maps $B_{r_0}(z_0)$ once around the remainder of S^2. Notice that $f|_{B_{r_0}(z_0)}$ is a convex function, so we would expect to see bubbling in the heat flow with initial data u_0.

As earlier in this thesis, our problem will be with bubbles forming at finite time. Let us make our one large assumption here: that *the bubble forms at infinite time.* As examined in chapter 5, the still-forming bubble

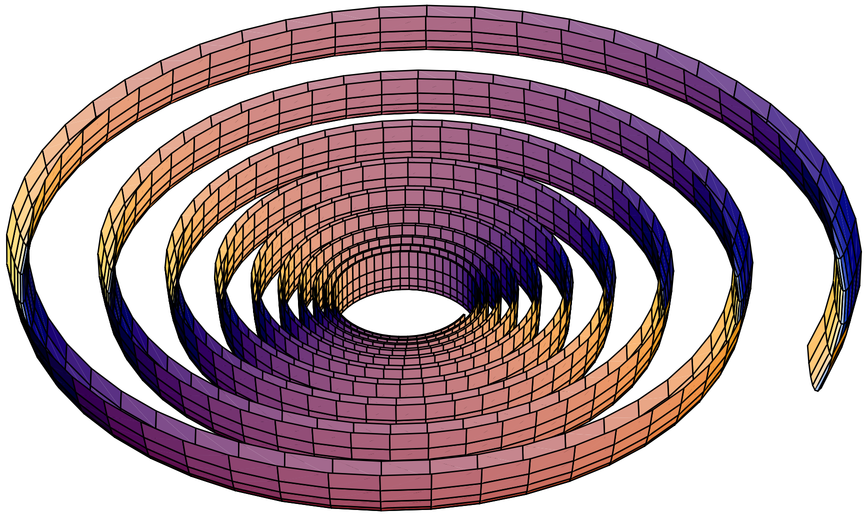

Figure 7.2: The parametric plot $\big((0.1 + 0.8e^{-\alpha s} + 0.1ye^{-\alpha s}(1 - e^{-\alpha}))\cos 2\pi s, (0.1 + 0.8e^{-\alpha s} + 0.1ye^{-\alpha s}(1 - e^{-\alpha}))\sin 2\pi s, 1 + \beta y^2 + \gamma e^{-\alpha s}\big)$ for $s \in [0, 10]$ and $y \in [-1, 1]$; showing the "record groove" $f : G_\alpha \to (0, \infty)$. (Diagram shows $\alpha = 0.3$, $\beta = 0.1$ and $\gamma = 0.2$.)

would "slide" down the slope of f. In this example, such a "forming bubble" would slide along the groove (with increasing s) indefinitely. Therefore, the distance between the centre of the forming bubble and the set $\partial B_{0.1}(0)$, would tend to zero as $t \to \infty$. Thus, any neighbourhood of any $z \in \partial B_{0.1}(0)$ would be "passed through" (infinitely many times) by the "forming bubble", so the derivative of u could not be uniformly bounded there.

Extended Summary

This is an extended form of the Abstract on page ix.

Chapter 1 - Introduction

Let $(\mathcal{M}, g)$ be a compact surface (with or without boundary) and let $(\mathcal{N}, h)$ be a compact Riemannian manifold without boundary, embedded isometrically in $\mathbb{R}^N$. Let $f : \mathcal{M} \to (0, \infty)$ be a smooth function. For $u \in W^{1,2}(\mathcal{M}; \mathcal{N})$ the *f-harmonic Energy* is

$$E_f(u) := \frac{1}{2} \int_{\mathcal{M}} f |\nabla u|^2 \, d\mathcal{M}.$$

(Weakly) f-harmonic maps are critical points of E_f under *admissible variations* $u_s = P \circ (u + s\phi)$ for $\phi \in C_c^\infty(\mathcal{M}; \mathbb{R}^N)$ and small s. Here P denotes nearest point projection onto $\mathcal{N}$ from a tubular neighbourhood of $\mathcal{N}$. (Classical) f-harmonic maps are C^2 maps satisfying this definition. Such maps satisfy the Euler-Lagrange equation

$$f \Delta u + f A(u)(\nabla u, \nabla u) + \nabla f * \nabla u = 0$$

where $\nabla f * \nabla u := \langle \nabla f, \nabla u^i \rangle \frac{\partial}{\partial u^i}$.

The *f-harmonic heat flow* is the solution to

$$\begin{cases} u_t - f(x) \Delta_{\mathcal{M}} u = f(x) A(u)(\nabla u, \nabla u) + \nabla f * \nabla u \\ u|_{t=0} = u_0 \\ u(\cdot, t)|_{\partial \mathcal{M}} = u_0|_{\partial \mathcal{M}}. \end{cases} \tag{1.5.1}$$

Working in implicit coordinates, the Euler-Lagrange equation is

$$\tau_f(u) := tr_g \nabla(f du) = 0$$

and the Jacobi Operator of an f-harmonic map is $J_{f,u} := -tr_g \nabla(f d^\nabla) -$

$fR^{\mathcal{N}}\big(\cdot, du\big)du$.

We only consider surfaces $\mathcal{M}$ for this reason: If $\dim \mathcal{M} \neq 2$, any f-harmonic map $(\mathcal{M}, g) \to (\mathcal{N}, h)$ would also be a harmonic map, by a conformal change of metric.

Chapter 2 - Properties of f-harmonic maps

Section 2.1 proves:

Theorem 2.1.5 (Puncture Repair). *Suppose that $u : D \setminus \{0\} \to \mathcal{N}$ is a smooth f-harmonic map with finite energy. Then u extends to a smooth f-harmonic map $u : D \to \mathcal{N}$.*

Then section 2.2 uses the Implicit Function Theorem to prove:

Proposition 2.2.2. *Suppose that $u \in C^\infty\big((\mathcal{M}, g); (\mathcal{N}, h)\big)$ is a non-constant f-harmonic map. Suppose further that there are no Jacobi Fields along u. Let $\alpha \in (0, 1)$. There exist $\delta > 0$ and $\varepsilon > 0$ such that; if $f_1 \in C^\infty(\mathcal{M}; (0, \infty))$ and $\|f_1 - f\|_{C^{1,\alpha}} < \delta$ then there exists a unique f_1-harmonic map $u_1 \in C^\infty(\mathcal{M}; \mathcal{N})$ such that $\|u_1 - u\|_{C^{2,\alpha}} < \varepsilon$.*

Chapter 3 - Heat Flow

We extend the existence and "bubbling" result of Struwe [Str96, Theorem III.6.6] to f-harmonic map heat flow:

Theorem 3.1.1 (f-harmonic Heat Flow). *Let $u_0 \in W^{1,2}(\mathcal{M}; \mathcal{N})$. If $\partial\mathcal{M}$ is non-empty, suppose further that $u_0|_{\partial\mathcal{M}} \in C^{2,\alpha}(\partial\mathcal{M}; \mathcal{N})$. There exists a weak solution $u : \mathcal{M} \times [0, \infty) \to \mathcal{N}$ of (1.5.1) with the following properties*

(i) *u is smooth on $\mathcal{M} \times (0, \infty)$ away from finitely many points $(\bar{x}_k, \bar{t}_k)$, $1 \leq k \leq K$, $0 < \bar{t}_k \leq \infty$;*

(ii) *$E_f\big(u(t)\big) \leq E_f\big(u(s)\big)$ for all $0 \leq s \leq t$; and*

(iii) *u assumes the initial data continuously in $W^{1,2}(\mathcal{M}, \mathcal{N})$.*

The solution u is unique in this class.

Furthermore, at a singular (or bubble) point $(\bar{x}, \bar{t}) \in \mathcal{M} \times (0, \infty]$, there exist sequences $x_m \to \bar{x}$, $t_m \nearrow \bar{t}$, $R_m \searrow 0$ and a non-constant harmonic map $\tilde{u} : \mathbb{R}^2 \to \mathcal{N}$ with finite (harmonic) energy, such that as $m \to \infty$,

$$u_m(x) := u\big(\exp_{x_m}(R_m x), t_m\big) \to \tilde{u}$$

in $W^{2,2}_{loc}(\mathbb{R}^2; \mathcal{N})$. Moreover $\tilde{u}$ extends to a smooth harmonic map $\bar{u} : \mathbb{R}^2 \cup \{\infty\} = S^2 \to \mathcal{N}$ which we call a 'bubble'.

There exists a further sequence of times $t_m \to \infty$ such that the sequence of maps $u(\cdot, t_m)$ converges weakly in $W^{1,2}(\mathcal{M};\mathcal{N})$ to a smooth f-harmonic map $u_\infty : \mathcal{M} \to \mathcal{N}$, and smoothly away from finitely many points $\bar{x}_k$.

Chapter 4 - Mapping the boundary to a point

A result of Lemaire (*"every harmonic map from a compact, contractible surface, with constant boundary data must be constant"*) is considered in chapter 4. The f-harmonic map heat flow yields examples demonstrating that this result doesn't extend completely to f-harmonic maps. There is however an analogue for certain f.

Proposition 4.1.1. *Suppose that $f : D \to (0, \infty)$ satisfies $\nabla f(x) \cdot x > 0$ for almost all $x \in D$. Then every smooth f-harmonic map $u \in C^\infty(D;\mathcal{N})$ which maps ∂D to a point p, is constant and takes the value p.*

The proof of this proposition requires an analysis of the Hopf Differential near the boundary of D.

Lemma 4.2.1. *There exist an f and a smooth non-constant, f-harmonic map from the disc to the 2-sphere which maps ∂D to a point.*

There is also a result by Eells-Wood [EW82] which states that: *There does not exist a harmonic map of degree 1, from the torus to the* 2*-sphere.* However:

Lemma 4.2.2. *There exist $f : T^2 \to (0, \infty)$ and a smooth degree* 1*, f-harmonic map from the square torus to the 2-sphere.*

A particular sequence of f_n-harmonic maps ($D \to S^2$, $\partial D \mapsto \{0\}$) is studied (see figure 4.4 on page 62), with $f_n \to 1$, to try to see how the f-harmonic map degenerates as the f is "flattened" towards $f \equiv 1$.

Chapter 5 - Bubbles sliding down hills

The main result of this thesis is the *Moving Bubble Theorem* which tells us that infinite time bubbles can only form at critical points of f.

Theorem 5.2.1 (Moving Bubble). *Let $\mathcal{M}$ be a smooth, compact, Riemannian surface, with or without with boundary. Let $z_0 \in \int \mathcal{M}$ and $u_0 \in W^{1,2}(\mathcal{M};\mathcal{N})$. Moreover if $\partial\mathcal{M}$ is non-empty, suppose further that $u_0|_{\partial\mathcal{M}} \in C^{2,\alpha}(\partial\mathcal{M};\mathcal{N})$. Suppose that the map $u : \mathcal{M} \times [0, \infty) \to \mathcal{N}$ solves*

$$\begin{cases} u_t - f(x)\Delta_{\mathcal{M}} u = f(x)A(u)(\nabla u, \nabla u) + \nabla f * \nabla u \\ u|_{t=0} = u_0 \\ u(\cdot, t)|_{\partial\mathcal{M}} = u_0|_{\partial\mathcal{M}}. \end{cases} \tag{1.5.1}$$

Suppose further that (z_0, ∞) *is a singular (or bubble) point of* u *– as described by Theorem 3.1.1. Then* $\nabla f(z_0) = 0$*.*

Immediately, we have

Corollary 5.2.1.1. *Let* $\mathcal{M}$ *be a smooth, compact, Riemannian surface, possibly with boundary. Suppose that* $u_0 \in W^{1,2}(\mathcal{M}; \mathcal{N})$*,* $u_0|_{\partial\mathcal{M}} \in C^{2,\alpha}(\partial\mathcal{M}; \mathcal{N})$ *and that the map* $u : \mathcal{M} \times [0, \infty) \to \mathcal{N}$ *solves* (1.5.1)*. Let* $\Omega \subset \operatorname{int} \mathcal{M}$ *be an open set such that* $\nabla f(x) \neq 0$ *for all* $x \in \overline{\Omega}$*. Then there exists a sequence* $t_n \to \infty$ *such that* $u(\cdot, t_n)|_\Omega$ *converges smoothly.*

Chapter 6 - Comments on the sharpness of the Moving Bubble Theorem

By taking an f without critical points, we consider a flow which must bubble in finite time.

Lemma 6.2.1. *There exists a* f*-harmonic heat flow, under which a finite time singularity can form at a non-critical point of* f*.*

A hypothesized refinement of the *Moving Bubble Theorem*, that infinite time bubbles only form at minima of f, is false.

Lemma 6.1.2. *Suppose that* $f : D \to (0, \infty)$ *is rotationally symmetric, and satisfies*

$$-\frac{\pi^4 r^3}{576} \leq \frac{f_r}{f} \leq 0.$$

Suppose that $u : D \times [0, \infty) \to S^2$ *solves the heat flow equation (1.5.1), where* $u_0(r, \phi) = (\phi, \pi r)$ *in polar coordinates. Then a bubble forms at the origin – the maximum of* f *– at infinite time.*

Chapter 7 - A suggested application of the Moving Bubble Theorem

Finally, we briefly describes a possible application of this f-harmonic map theory.

Notation

Presented below is a quick reference to a selection of the notation used in this work.

$W^{k,p}(\mathcal{M},\mathcal{N})$	the Sobolev space of all maps $\mathcal{M} \to \mathcal{N}$ for which all derivatives up to order k are elements of $L^p(\mathcal{M},\mathcal{N})$.
$C_c^\infty(\mathcal{M},\mathcal{N})$	the set of smooth maps with compact support.
$W_0^{k,p}(\mathcal{M},\mathcal{N})$	the closure of $C_c^\infty(\mathcal{M};\mathcal{N})$ in the Sobolev space $W^{k,p}(\mathcal{M},\mathcal{N})$.
$C^{k,\alpha}(\mathcal{M},\mathcal{N})$	space of all maps $\mathcal{M} \to \mathcal{N}$ which have α-Hölder continuous derivatives up to order k.
$\mu^{r,\alpha}(\mathcal{M})$	space of all Riemannian metrics on $\mathcal{M}$ which have α-Hölder continuous derivatives up to order r. (page 20)

$$E_f(u) = \frac{1}{2}\int_{\mathcal{M}} f|\nabla u|^2 \, d\mathcal{M}, \qquad \text{(page 1)}$$

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}, \qquad e_i = \frac{\partial}{\partial u^i}, \qquad \partial_t = \frac{\partial}{\partial t},$$

$$\nabla f * \nabla u = \left\langle \nabla f, \nabla u^i \right\rangle e_i,$$

A the second fundamental form of the embedding of $\mathcal{N}$ in $\mathbb{R}^N$.

$$D_\rho = B_\rho(0) \subset \mathbb{R}^2,$$

$$V_\rho\mathcal{N} = \{z \in \mathbb{R}^N : d(z,\mathcal{N}) < \rho\}, \qquad \text{(page 21)}$$

$$U_y = \{z \in V_\rho\mathcal{N} : P(z) = y\}, \qquad \text{(page 21)}$$

$$U_\alpha(x) = \left(\frac{x}{|x|}\sin\alpha(|x|), \cos\alpha(|x|)\right), \qquad \text{(page 58)}$$

$\mathbb{U}^2$ the upper-half-plane $\{(x,y) \in \mathbb{R}^2 : y \geq 0\}$

$\sqrt{g}$ the square root of the determinant of g.

$$\Delta_{\mathcal{M}} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\right),$$

$$d\mathcal{M} = \sqrt{g}dx,$$

$\imath_{\mathcal{M}}$ the injectivity radius of the exponential map on $\mathcal{M}$. (page 31)

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